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Twisted Product and Cohomology

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Abstract Let H be a Hopf algebra, H_1 be a sub-Hopf algebra of H, H_2 be the quotient Hopt algebra of H modular H_1. This paper gives a simplified complex by defining a new base for the cobar complex and proves that the cobar complex of H has the same cohomology algebra with a twisted product of the cobar complexes of H_1 and H_2.

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1 Introduction

Let H be a Hopf algebra, H_1 be a sub-Hopf algebra of H, H_2 be the quotient Hopf algebra of H modulo H_1. Cartan and Eilenberg proved[3] that there exists a spectral sequence $\mathbf{E}^r_{n,m,t}$ such that $\mathbf{E}^2 = \mathrm{H}^{*,*}(H_1) \otimes \mathrm{H}^{*,*}(H_2)$. Since the structure of the cobar complex is very complicated, the computing of this spectral sequence remains almost unsolved. This paper gives a simplified complex by defining a new base for the cobar complex. The idea is inspired by Zhou's paper[6] and I would like to express my deep gratitude to him. I must also thank Prof. Lin Jinkun for his many useful advices.

In this paper, K is a field, all algebraic structures considered are vector spaces over K, and most of the results of this paper can not be naturally generalized to the case when K is a ring. All the algebras in this paper are non-negatively graded (or bigraded) and there is a finite dimensional vector space at a fixed degree, so all the elements or bases of a vector space are homogenous. Notice that for a commutative graded algebra $xy = (-1)^{\|x\|\|y\|}yx$ for any $x, y \in R$, and that for a bigraded algebra, $xy = (-1)^{|x||y|+\|x\|\|y\|}yx$, where we always use $|\cdot|$ to denote the cohomological degree and $\|\cdot\|$ to denote the second algebraic degree. In this paper, all the complexes (mainly DGA) are bigraded and all definitions cited here may be seen in the references.

2 DGA's and Their Products

We call a chain complex (R, d) ($d : R^{s,t} \longrightarrow R^{s+1,t}$) a DGA if R is an algebra with unit and for any $x, y \in R$, $dxy = (dx)y + (-1)^{|x|}x(dy)$.

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Let Λ be an index set, let $F(\{x_\alpha\}_{\alpha\in\Lambda})$ be the algebra with unit freely generated by the set $\{x_\alpha\}_{\alpha\in\Lambda}$, that is, it has a base

$$\{1\}\cup\{x_{\alpha_1}x_{\alpha_2}\cdots x_{\alpha_n}\mid \alpha_i\in\Lambda,\ i{=}1,\cdots,n,\ n{=}1,2,\cdots\}.$$

A DGA (R,d) is called free if there is an index set Λ such that for any $\alpha\in\Lambda$, there is an $x_\alpha\in\bar{R}$ and $R{=}F(\{x_\alpha\}_{\alpha\in\Lambda})$; the set $\{x_\alpha\}_{\alpha\in\Lambda}$ is called the set of generators of (R,d). Notice that for a given free DGA, there may be more than one set of generators, but when we are given a free DGA, we always take a fixed set of generators.

The cobar complex of a coalgebra is just a free algebra. Let V be a coalgebra, Δ be its diagonal map and ε its augmentation, $\bar{V}=\ker\varepsilon$, let

$$C^{s,*}=\underbrace{\bar{V}\otimes_K\bar{V}\otimes_K\cdots\otimes_K\bar{V}}_{s-fold}\qquad C(V)=K\oplus\sum_{s=1}^{\infty}C^s(V).$$

Then $\otimes_K$ makes $C(V)$ an algebra; we use "|" to denote the product of $C(V)$ and use $[a_1|a_2|\cdots|a_n]$, $a_i\in\bar{V}$ to denote the element of $C^s(V)$. Then if $\{x_\alpha\}_{\alpha\in\Lambda}$ is a base for $\bar{V}$, $\{[x_\alpha]\}_{\alpha\in\Lambda}$ is a set of generators for $C(V)$, and $d[x_\alpha]=-\sum[x'_\alpha|x''_\alpha]$, where $\Delta(x_\alpha)=1\otimes x_\alpha+x_\alpha\otimes 1+\sum x'_\alpha\otimes x''_\alpha$, so $(C(V),d)$ is a free DGA.

Let (R_1,d_1), (R_2,d_2) be DGA's, $(R_1\otimes R_2,d_1\otimes d_2)$ have a natural definition. Now we will define the $*$-product. If $\{x_\alpha\}_{\alpha\in\Lambda}$, and $\{y_\beta\}_{\beta\in\Pi}$ are respectively the sets of generators of R_1 and R_2, $\{p_{\alpha'}\}_{\alpha'\in\Lambda'}$, $\{q_{\beta'}\}_{\beta'\in\Pi'}$ are respectively the sets of zero relations, then the algebra R_1*R_2 is generated by the set $\{x_\alpha\}_{\alpha\in\Lambda}\cup\{y_\beta\}_{\beta\in\Pi}$ with the set of zero relations $\{p_{\alpha'}\}_{\alpha'\in\Lambda'}\cup\{q_{\beta'}\}_{\beta'\in\Pi'}$; it is easy to check that R_1*R_2 is independent of the choice of the base and that R_1 and R_2 are all subalgebras of R_1*R_2 and d_1 and d_2 have a unique extension on R_1*R_2. Moreover, we have (see [5]) $\mathrm{H}^{*,*}(R_1*R_2)=\mathrm{H}^{*,*}(R_1)*\mathrm{H}^{*,*}(R_2)$.

Let $\{\mathbf{F}_n(R)\}$ be a filtration of (R,d). We call it a $\mathcal{D}$-filtration if for any $s,t\geqslant 0$, $\mathbf{F}_s(R)\mathbf{F}_t(R)\subset\mathbf{F}_{s+t}(R)$.

Definition 2.1 *Let $\{\mathbf{F}_n(R)\}$ be a $\mathcal{D}$-filtration of R, define*

$$(\mathbf{F}^0_{n,s,t},\tilde{d})=(\mathbf{F}_n(R)^{s,t}/\mathbf{F}_{n-1}(R)^{s,t},\tilde{d}).$$

Then $(\mathbf{F^0},\tilde{d})$ (regardless of the new gradation) is a DGA, its product is defined by

$$[x]_n[y]_m=\begin{cases}[xy]_{n+m}\in\mathbf{F}^0_{n+m,|x|+|y|,\|x\|+\|y\|}, & \text{if } xy\notin\mathbf{F}_{n+m-1}(R),\\ 0, & \text{if } xy\in\mathbf{F}_{n+m-1}(R),\end{cases}$$

where $x\in\mathbf{F}_n(R)-\mathbf{F}_{n-1}(R)$, $y\in\mathbf{F}_m(R)-\mathbf{F}_{m-1}(R)$, $[\quad]_k$ denotes the quotient image in $\mathbf{F}_k(R)/\mathbf{F}_{k-1}(R)$.

Notice that for any filtration $\{\mathbf{F}_n(R)\}$, there is a spectral sequence $\mathbf{F}^r_{n,s,t}$ $(r\geqslant 1)$ converging to $\mathrm{H}^{*,*}(R)$, but when $\{\mathbf{F}_n(R)\}$ is a $\mathcal{D}$-filtration, $\mathrm{H}^{*,*}(\mathbf{F}^0)=\mathrm{H}^{*,*}(\mathbf{F}^1)$, and all $(\mathbf{F}^r,d^r)$ are DGA's (regardless of the new gradation).

Definition 2.2 *Let (R_1,d_1), (R_2,d_2) be DGA's. (R,d) is called a twisted product of (R_1,d_1) with (R_2,d_2) if there exists a $\mathcal{D}$-filtration $\{\mathbf{F}_n(R)\}$ such that $\left(\mathbf{F}^0(R),\tilde{d}\right){=}(R_1{\otimes}R_2,d_1{\otimes}d_2)$, where "=" is both a DGA and a trigraded algebra isomorphism and $R_1\otimes R_2$ is trigraded by*

$$x\otimes y\in(R_1\otimes R_2)^{\|y\|,|x|+|y|,\|x\|+\|y\|}$$

for $x \in R_1, y \in R_2$.

We denote a given twisted product by $(R_1 \rtimes R_2, d_1 \rtimes d_2)$. The tensor product is itself a twisted product, but there exists more complicated twisted product. By definition, (R_1, d_1) is always a sub-DGA of $(R_1 \rtimes R_2, d_1 \rtimes d_2)$. If (R_2, d_2) is free, then we may regard R_2 as a subalgebra of $(R_1 \rtimes R_2, d_1 \rtimes d_2)$; the reason is as follows. Suppose $\{u_\alpha\}_{\alpha\in\Lambda}$ is a set of generators of R_2, $\{x_\alpha\}_{\alpha\in\Lambda}$ is a subset of $R_1 \rtimes R_2$ such that $[x_\alpha] = u_\alpha$ for all $\alpha \in \Lambda$,where [ ] denotes the quotient image. Then by definition,

$$[x_{\alpha_1} x_{\alpha_2} \cdots x_{\alpha_n}] = u_{\alpha_1} u_{\alpha_2} \cdots u_{\alpha_n},$$

so $F(\{x_\alpha\}_{\alpha\in\Lambda})$ is an algebra isomorphic to R_2.

Definition 2.3 *A DGA* (R, d) *with the set of generators* $\{x_\alpha\}_{\alpha\in\Lambda}$ *is called minimal if* $|x_\alpha| = 1$ *for all* $\alpha \in \Lambda$.

Let V be a coalgebra. Then $(C(V), d)$ is a minimal DGA.

Before we state the central theorem of this paper, we give the conditions of the theorem. Let $(R_1, d_1), (R_2, d_2)$ be two minimal DGA's with respectively the sets of generators $\{x_\alpha\}_{\alpha\in\Lambda}, \{y_\beta\}_{\beta\in\Pi}$, (R, d) be a minimal DGA with the set of generators

$$\{x_\alpha\}_{\alpha\in\Lambda} \cup \{y_\beta\}_{\beta\in\Pi} \cup \{z_\gamma\}_{\gamma\in\Lambda\times\Pi}.$$

We call these generators of R pure letters; the finite product of pure letters is called a pure word, then all pure words form a base for R. Now we will give a new base for R. For any $\gamma \in \Lambda \times \Pi$, we call dz_γ a mixed letter, the finite product of mixed letters or the finite product of mixed letters and pure letters are called a mixed word. A word is called standard if it has no factor $y_\beta x_\alpha$ for all $(\alpha, \beta) \in \Lambda \times \Pi$. Under the conditions of the next theorem, all standard words form a new base for R. We define the length of a word (mixed or pure) to be

$$\|b\|_1 = \begin{cases} \|y_\beta\|, & \text{if } b = y_\beta \text{ for some } \beta \in \Pi, \\ 0, & \text{otherwise.} \end{cases}$$

If $b = b_1 b_2 \cdots b_n$,b_i are letters, then define $\|b\|_p = \sum_{i=1}^n \|b_i\|_1$.

Theorem 2.1 $(R_1, d_1), (R_2, d_2), (R, d)$ *are as above and satisfy*

1. (R_1, d_1) *is a sub-DGA of* (R, d),

2. *For any* $\beta \in \Pi$,*there exist pure words* u_i *and* $k_i \in K$, *such that* $u_i \notin R_2, \|u_i\|_p < \|y_\beta\|$, *and*

$$dy_\beta = d_2 y_\beta + \sum_i k_i u_i,$$

3. *For any* $(\alpha, \beta) \in \Lambda \times \Pi$,*there exist pure words* v_j *and* $k_j \in K$ *such that* $v_j \notin R_2, \|v_j\|_p < \|y_\beta\|$, *and*

$$dz_{\alpha,\beta} = -x_\alpha y_\beta - (-1)^{\|x_\alpha\|\|y_\beta\|} y_\beta x_\alpha + \sum_j k_j v_j.$$

Let I *be the* $\mathcal{D}$*-ideal of* (R, d) *generated by all* z_γ *and* $dz_\gamma, \gamma \in \Lambda \times \Pi$, $q\colon R \to R/I$ *is the quotient map. Then* q *induces an isomorphism between the cohomology algebras, and* $(R/I, \tilde{d})$ *is a twisted product of* (R_1, d_1) *with* (R_2, d_2).

Proof It is easy to prove that $d(I) \subset I$, so $(R/I, \tilde{d})$ is still a DGA. By the conditions of the theorem we can prove (see [5]) that all standard words form a base for R, and for any word

b, if the linear combination of standard words for b is $b=\sum_{i=1}^{n} k_i b_i$, then we have $\|b\|_p \geqslant \|b_i\|_p$ for $i=1,2,\cdots,n$.

Let $\{\mathbf{F}_n(R)\}$ be the subvector space of R spanned by all standard words b such that $\|b\|_p \leqslant n, n=1,2,\cdots$. Then it is easy to prove that $\{\mathbf{F}_n(R)\}$ is a $\mathcal{D}$-filtration of (R,d). Now consider the structure of $\left(\mathbf{F}^0,\tilde{d}\right)$, and use $[\quad]$ to denote the quotient image. Then

$$\begin{aligned}
&\tilde{d}[x_\alpha]=[d_1x_\alpha], && \text{for all } \alpha\in\Lambda[x_\alpha]\in\mathbf{F}^0_{0,1,\|x_\alpha\|}(R),\\
&\tilde{d}[y_\beta]=[dy_\beta]=[d_2y_\beta], && \text{for all } \beta\in\Pi,[y_\beta]\in\mathbf{F}^0_{\|y_\beta\|,1,\|y_\beta\|}(R),\\
&\tilde{d}[z_{\alpha,\beta}]=[dz_{\alpha,\beta}], && \text{for all } (\alpha,\beta)\in\Lambda\times\Pi,[z_{\alpha,\beta}]\in\mathbf{F}^0_{0,1,\|z_{\alpha,\beta}\|}(R),\\
&\tilde{d}[dz_{\alpha,\beta}]=0, && \text{for all } (\alpha,\beta)\in\Lambda\times\Pi,[c_{\alpha,\beta}]\in\mathbf{F}^0_{0,2,\|c_{\alpha,\beta}\|}(R),
\end{aligned}$$

$$[x_\alpha][y_\beta]=[x_\alpha y_\beta]=(-1)^{1+\|x_\alpha\|\|y_\beta\|}[y_\beta][x_\alpha].$$

Let $(R_3,d_3)\overset{\mathcal{D}}{=}\left(F(\{[z_\gamma],[dz_\gamma]\}_{\gamma\in\Lambda\times\Pi}),d^0\right)$. Then

$$\left(\mathbf{F}^0(R),\tilde{d}\right)=\left((R_1\otimes R_2)*R_3,(d_1\otimes d_2)*d_3\right),$$

$$\mathbf{F}^1_{*,*,*}(R)=\mathrm{H}^{*,*}(R_1)\otimes\mathrm{H}^{*,*}(R_2).$$

Since the restriction of $\{\mathbf{F}_n(R)\}$ on I is also a $\mathcal{D}$-filtration of I, and an easy computation shows that $(\mathbf{F}^0(I),\tilde{d})$=$(R_3,d_3)$, so $\mathrm{H}^{*,*}(I)=0$, q induces an isomorphism of cohomology algebras, and it is also easy to prove that the quotient DGA $(R/I,\tilde{d})$ has a $\mathcal{D}$-filtration $\{\mathbf{F}_n(R/I)\}=\{q(\mathbf{F}_n(R))\}$ such that

$$\left(\mathbf{F}^0(R/I),\tilde{d}\right)=(R_1\otimes R_2,d_1\otimes d_2),$$

so $(R/I,\tilde{d})$ is a twisted product.

Now let us consider the structure of $(R_1\rtimes R_2,d_1\rtimes d_2)$. Let $\{x_\alpha\}_{\alpha\in\Lambda}$, $\{y_\beta\}_{\beta\in\Pi}$ be the sets of generators of R_1 and R_2. Then $R_1\rtimes R_2$=$R_1\otimes_K R_2$, where "=" is a vector space isomorphism but not an algebraic one, so we may trigraded $(R_1\rtimes R_2,d_1\rtimes d_2)$ through this isomorphism as in Definition 2.2. Then for any $\alpha\in\Lambda$, $\beta\in\Pi$,

$$(d_1\rtimes d_2)x_\alpha=d_1x_\alpha,$$

$$(d_1\rtimes d_2)y_\beta=d_2y_\beta+\sum z_i,$$

$$x_\alpha y_\beta-(-1)^{|x_\alpha||y_\beta|+\|x_\alpha\|\|y_\beta\|}y_\beta x_\alpha=\sum z_j,$$

where the new gradations of z_i and z_j are all less than $<\|y_\beta\|$.

3 Applications to the Cohomology of Hopf Algebras

Definition 3.1 *Let R_1,R_2 be two graded algebras. If we change "bigraded" in Definition 2.2 to "graded" and take d_1=0, d_2=0, then we get the definition of the twisted product of R_1 with R_2, and for two coalgebras V_1, V_2, we define*

$$V_1\rtimes V_2=((V_2^*\rtimes V_1)^*)^*$$

where $*$ denotes the dual space.

Theorem 3.1 *Let $V_1 \rtimes V_2$ be a given twisted product. Then there are a twisted product $(C(V_1)\rtimes C(V_2)), d_1 \rtimes d_1)$ and a quotient map $q\colon C(V_1 \rtimes V_2) \to C(V_1)\rtimes C(V_2)$ such that q induces an isomorphism between the cohomology algebras. If $V_1 \rtimes V_2 = V_1 \otimes V_2$, the quotient DGA is also a tensor product of DGA's.*

Proof Let $\{y_\beta\}_{\beta\in\Pi}$ be a base for $\overline{V_2^*}$. $\{x_\alpha\}_{\alpha\in\Lambda}$ is a subset of $V_2^* \rtimes V_1^*$ such that $\{[x_\alpha]\}_{\alpha\in\Lambda}$ is a base for $\overline{V_1^*} \subset \mathbf{F}^0(V_2^* \rtimes V_1^*)$, where [ ] denotes the quotient image in $\mathbf{F}^0(V_2^* \rtimes V_1^*)$. Then $\overline{V_2^* \rtimes V_1^*}$ has a base

$$\{x_\alpha\}_{\alpha\in\Lambda} \cup \{y_\beta\}_{\beta\in\Pi} \cup \{x_\alpha y_\beta\}_{(\alpha,\beta)\in\Lambda\times\Pi}.$$

It is easy to check that $[x_\alpha^*]$, $[y_\beta^*]$, $[(x_\alpha y_\beta)^*]$ as the elements of cobar complexes ([ ] is not the quotient image) corresponding to x_α,y_β,$z_{\alpha,\beta}$ in Theorem 2.1 satisfy the conditions of the theorem. If $V_1 \rtimes V_2 = V_1 \otimes V_2$, then the spectral sequence collapses from $\mathbf{F}^1$.

For commutative, coassociative Hopf algebras, the theorem is more obvious. Let H_1 be a sub-Hopf algebra of H, I be the ideal of H generated by H_1. Then $H_2 = H/I$ has a unique induced Hopf algebra structure, and we denote it by H/H_1.

Theorem 3.2 *Suppose H_1 is a sub-Hopf algebra of H, $H_2 = H/H_1$. Then there is a quotient map*

$$q\colon (C(H), d) \to (C(H_1) \rtimes C(H_2), d_1 \rtimes d_2)$$

that induces an isomorphism between the cohomology algebras; and if we have $H = H_1 \otimes H_2$, then quotient DGA is $(C(H_1) \otimes C(H_2), d_1 \otimes d_2)$.

Proof By checking directly.

Now let K be the field of integers modulo an odd prime p, $H = P(\xi)$, $\|\xi\|$ be an even integer, P be the cohomology algebra, ξ be primitive. Then it is easy to check that $V_i = P(\xi^{p^i}/(\xi^{p^{i+1}})$ $(i = 0, 1, \cdots)$ is a subcoalgebra of H, and it is easy to compute that $H^{*,*}(V_i) = P(b_i) \otimes E(h_i)$, where $b_i = \sum_{j=1}^{p-1} \binom{p}{j}/p[\xi^{jp^i}|\xi^{(p-j)p^j}]$, $h_i = [\xi^{p^i}]$, since as an coalgebra we have $H = \otimes_{i=1}^{\infty} V_i$; so we have

$$H^{*,*} = P(b_0, b_1, \cdots) \otimes E(h_0, h_1, \cdots).$$

When incorporating the computation of the multi-product $<a; b_1, \cdots, b_n>$, we may compute even more complicated Hopf algebras (see [5]).

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