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Triangular Matrix Representations of Rings of Generalized Power Series

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Abstract Let R be a ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S. If either $(S, \leq)$ satisfies the condition that $0 \leq s$ for all $s \in S$, or R is reduced, then the ring $[[R^{S,\leq}]]$ of the generalized power series with coefficients in R and exponents in S has the same triangulating dimension as R. Furthermore, if R is a PWP ring, then so is $[[R^{S,\leq}]]$.

Keywords Generalized triangular matrix representation, Twisted generalized power series ring, PWP ring, Triangulating dimension
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1 Introduction

All rings considered here are associative with identity and R denotes such a ring. Any concept and notation not defined here can be found in [1, 2].

Recall from [3, 4] that an idempotent $e \in R$ is *left* (resp. *right*) *semicentral* in R if $ere = re$ (resp. $ere = er$), for all $r \in R$. Equivalently, $e^2 = e \in R$ is left (resp. right) semicentral if eR (resp. Re) is an ideal of R. We use $\mathscr{S}_l(R)$ and $\mathscr{S}_r(R)$ for the sets of all left and all right semicentral idempotents of R, respectively. From [5], an idempotent e of R is called *semicentral reduced* if $\mathscr{S}_l(eRe) = \{0, e\}$. A ring R is called *semicentral reduced* [5, 6] if 1 is semicentral reduced. From [5] a ring R has a *generalized triangular matrix representation* if there exists a ring isomorphism

$$\theta : R \longrightarrow \begin{pmatrix} R_1 & R_{12} & \cdots & R_{1n} \\ 0 & R_2 & \cdots & R_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & R_n \end{pmatrix},$$

where each diagonal ring, R_i, is a ring with unity, R_{ij} is a left R_i-right R_j-bimodule for $i < j$, and the matrices obey the usual rules for matrix addition and multiplication. If each R_i is semicentral reduced, then R has a *complete generalized triangular matrix representation* with triangulating dimension n ([5, 7]).

Recall from [3, 5, 7] that a *piecewise prime ring* (PWP ring for short) is a quasi-Baer ring with finite triangulating dimension. In [5, Corollary 4.13] it was shown that the class of PWP rings properly includes all piecewise domains which were introduced in [8] (hence all right hereditary rings which are semiprimary or right Noetherian). Every PWP ring has a complete generalized triangular matrix representation with prime diagonal rings, R_i, (see [5, Theorem 4.4]). It was observed in [8, p. 554] that n-by-n matrix rings and polynomial rings over piecewise domains are again piecewise domains. In [7], Birkenmeier and Keol Park showed

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that for a PWP ring R the following ring extensions are PWP rings: $R[G]$, the monoid ring of a u.p.-monoid G; $R[X]$ and $R[[X]]$, where X is a nonempty set of not necessarily commuting indeterminates; $R[x,x^{-1}]$ and $R[[x,x^{-1}]]$, the Laurent polynomial ring and Laurent series ring, respectively; $R[x;\alpha]$ and $R[[x;\alpha]]$, the skew polynomial and skew power series ring, respectively, where α is a particular type of ring automorphism of R; $T_n(R)$ and $Mat_n(R)$ the n-by-n upper triangular and full matrix rings over R, respectively. Also the open problems raised in [7] to enlarge the class of ring extensions of PWP rings which are also PWP rings and to enlarge the class of ring extensions of rings with finite triangulating dimension which also have finite triangulating dimension. In this paper we will show that, under some additional conditions, the ring $[[R^{S,\leq}]]$ of generalized power series with coefficients in R and exponents in S has the same triangulating dimension as R. Furthermore, if R is a PWP ring, then so is $[[R^{S,\leq}]]$, hence $[[R^{S,\leq}]]$ has a complete generalized triangular matrix representation with prime diagonal rings. In fact, we work for a more general ring extension which is called the twisted generalized power series rings introduced in Section 2. Thus our results hold for twisted generalized power series rings $[[R^{S,\leq},\lambda]]$.

2 Twisted Generalized Power Series Rings

Let $(S,\leq)$ be an ordered set. Recall that $(S,\leq)$ is *artinian* if every strictly decreasing sequence of elements of S is finite, and that $(S,\leq)$ is *narrow* if every subset of pairwise order-incomparable elements of S is finite. *Henceforth S always denotes a commutative monoid.* Unless stated otherwise, the operation of S shall be denoted additively, and the neutral element by 0.

Let $(S,\leq)$ be a strictly ordered monoid (that is, $(S,\leq)$ is an ordered monoid satisfying the condition that if $s,s',t\in S$ and $s<s'$, then $s+t<s'+t$), and R a ring. Let $[[R^{S,\leq}]]$ be the set of all maps $f:S\longrightarrow R$ such that $\mathrm{supp}(f)=\{s\in S|f(s)\neq 0\}$ is artinian and narrow. For any $s\in S$ and any $f_1,f_2,\ldots,f_n\in[[R^{S,\leq}]]$, denote $X_s(f_1,f_2,\ldots,f_n)=\{(u_1,u_2,\ldots,u_n)\in S^n|s=u_1+u_2+\cdots+u_n, f_1(u_1)\neq 0, f_2(u_2)\neq 0,\ldots,f_n(u_n)\neq 0\}$. The following result appeared in [2, 4.1].

Lemma 2.1 *$X_s(f_1,f_2,\ldots,f_n)$ is a finite set.*

Denote by $R\,\mathrm{End}(R)$ the set of all ring homomorphisms from R to R. Let $\lambda: S\longrightarrow R\,\mathrm{End}(R)$ be a map satisfying the following condition:

$$\lambda(u+v)=\lambda(u)\lambda(v),\qquad \forall\, u,v\in S.$$

For any $f,g\in[[R^{S,\leq}]]$, define $fg:S\longrightarrow[[R^{S,\leq}]]$ via

$$(fg)(s)=\sum_{(u,v)\in X_s(f,g)} f(u)\lambda(u)(g(v)).$$

Note that there are only finitely many non-zero summands. It is easy to see that $\mathrm{supp}(fg)\subseteq\mathrm{supp}(f)+\mathrm{supp}(g)$. Thus by [2, 2.1], $\mathrm{supp}(fg)$ is artinian and narrow, hence $fg\in[[R^{S,\leq}]]$. This defines a binary operation of multiplication on $[[R^{S,\leq}]]$.

For any $f,g,h\in[[R^{S,\leq}]]$ and any $s\in S$,

$$\begin{aligned}(f(gh))(s)&=\sum_{(u,u')\in X_s(f,gh)} f(u)\lambda(u)((gh)(u'))\\&=\sum_{(u,u')\in X_s(f,gh)} f(u)\lambda(u)\Big(\sum_{(v,w)\in X_{u'}(g,h)} g(v)\lambda(v)(h(w))\Big)\\&=\sum_{(u,u')\in X_s(f,gh)} f(u)\Big(\sum_{(v,w)\in X_{u'}(g,h)} \lambda(u)(g(v))\lambda(u)(\lambda(v)(h(w)))\Big)\\&=\sum_{(u,u')\in X_s(f,gh)} f(u)\Big(\sum_{(v,w)\in X_{u'}(g,h)} \lambda(u)(g(v))\lambda(u+v)(h(w))\Big)\end{aligned}$$

$$= \sum_{(u,u')\in X_s(f,gh)} f(u)\Bigg(\sum_{(v,w)\in X_{u'}(g,h)} \lambda(u)(g(v))\lambda(u+v)(h(w))\Bigg)$$
$$+ \sum_{(u,v,w)\in X} f(u)\lambda(u)(g(v))\lambda(u+v)(h(w))$$
$$= \sum_{(u,v,w)\in X_s(f,g,h)} f(u)\lambda(u)(g(v))\lambda(u+v)(h(w)),$$

where $X = \{(u,v,w)|(u,v,w) \in X_s(f,g,h), (gh)(v+w) = 0\}$. On the other hand,

$$((fg)h)(s) = \sum_{(w',w)\in X_s(fg,h)} (fg)(w')\lambda(w')(h(w))$$
$$= \sum_{(w',w)\in X_s(fg,h)} \Bigg(\sum_{(u,v)\in X_{w'}(f,g)} f(u)\lambda(u)(g(v))\Bigg)\lambda(w')(h(w))$$
$$= \sum_{(w',w)\in X_s(fg,h)} \sum_{(u,v)\in X_{w'}(f,g)} f(u)\lambda(u)(g(v))\lambda(u+v)(h(w))$$
$$= \sum_{(u,v,w)\in X_s(f,g,h)} f(u)\lambda(u)(g(v))\lambda(u+v)(h(w)).$$

Thus $(fg)h = f(gh)$. Now, with pointwise addition, and the multiplication as above, it is easy to see that $([[R^{S,\leq}]], +, \cdot)$ becomes a ring, which we denote by $[[R^{S,\leq}, \lambda]]$, and call the *twisted generalized power series ring related to* λ. The elements of $[[R^{S,\leq}, \lambda]]$ are called *generalized power series with coefficients in* R *and exponents in* S.

Example 2.2 1 If $\lambda(s) = 1$, the identity map of R, for every $s \in S$, then $[[R^{S,\leq}, \lambda]] = [[R^{S,\leq}]]$ is the ring of generalized power series in the sense of Ribenboim [1, 2]. Thus, the following rings are twisted generalized power series rings: the monoid ring $R[S]$ of S with coefficients in R; the group ring $R[G]$ of Abelian group G with coefficients in R; the ring $R[[x_1, \ldots, x_n]]$ of formal power series in n indeterminates and coefficients in R; Laurent series ring $R[[x, x^{-1}]]$; the ring $[[R^{\mathbb{N}_{\geq 1},\leq}]]$ of arithmetical functions with values in R, endowed with the Dirichlet convolution:

$$(fg)(n) = \sum_{d|n} f(d)g(n/d), \qquad \text{for each} \qquad n \geq 1,$$

where $\leq$ is the usual order of $\mathbb{N}$. Further work on the rings of generalized power series appears in [1, 2, 9, 10, 11].

2 Let α be a ring endomorphism of R. Let $S = \mathbb{N} \cup \{0\}$ be endowed with the usual order, and define $\lambda : S \longrightarrow R\,\mathrm{End}(R)$ via $\lambda(0) = 1$, the identity map of R, and $\lambda(k) = \alpha^k$ for any $k \in \mathbb{N}$. Then $[[R^{S,\leq}, \lambda]] = R[[x; \alpha]]$, the usual skew power series rings.

3 Let α be a ring automorphism of R. Let $S = \mathbb{Z}$ be endowed with the usual order, and define $\lambda : S \longrightarrow R\,\mathrm{End}(R)$ via $\lambda(s) = \alpha^s$. Then $[[R^{S,\leq}, \lambda]] = R[[x, x^{-1}; \alpha]]$, the usual skew Laurent series rings.

4 Let R be a ring and let G be an Abelian group acting on R as a group of automorphisms. Define $\lambda : G \longrightarrow R\,\mathrm{End}(R)$ via $\lambda(s) = s$ for any $s \in G$. Let $\leq$ be the trivial order of G. Then it is easy to see that $[[R^{G,\leq}, \lambda]] = R * G$, the skew group ring of G with coefficients in R. If G is an infinite cyclic group generated by σ where σ acts on R as a ring automorphism, then $[[R^{G,\leq}, \lambda]]$ is isomorphic to the skew Laurent polynomial ring $R[x, x^{-1}; \sigma]$.

5 Let R be a ring and α a ring homomorphism of R. Set $S = \mathbb{N} \cup \{0\}$ endowed with the trivial order. Define $\lambda : S \longrightarrow R\,\mathrm{End}(R)$ via $\lambda(0) = 1$, the identity map of R, and $\lambda(s) = \alpha^s$ for any $s \in \mathbb{N}$. Then $[[R^{S,\leq}, \lambda]] \cong R[x; \alpha]$, the usual skew polynomial ring.

6 Let R be the complex field and $0 \neq q \in R$. Let α be the R-automorphism on $R[x]$ determined by $\alpha(x) = qx$. Define $\lambda : \mathbb{N} \cup \{0\} \longrightarrow R\,\mathrm{End}(R[x])$ via $\lambda(0) = 1$, the identity

map of $R[x]$, and $\lambda(k) = \alpha^k$ for any $k \in \mathbb{N}$. Let $\leq$ be the trivial order over $\mathbb{N} \cup \{0\}$. Then $[[(R[x])^{\mathbb{N}\cup\{0\},\leq}, \lambda]] \cong R[x][y;\alpha]$, the quantum plane [4, 12].

7 Let F be a field. Let α be the F-automorphism of $F[x]$ sending x to $x-1$. Define $\lambda : \mathbb{N} \cup \{0\} \longrightarrow R\,\mathrm{End}(F[x])$ via $\lambda(0) = 1$, the identity map of $F[x]$, and $\lambda(k) = \alpha^k$ for any $k \in \mathbb{N}$. Let $\leq$ be the trivial order over $\mathbb{N} \cup \{0\}$. If V is the binary space $Fe_1 \oplus Fe_2$ with a Lie algebra structure given by the Lie product $[e_1, e_2] = e_2$, then $[[(F[x])^{\mathbb{N}\cup\{0\},\leq}, \lambda]] \cong F[x][y;\alpha]$, the universal enveloping algebra U of $(V, [,])$.

8 Let α and β be ring endomorphisms (respectively, ring automorphisms) of R such that $\alpha\beta = \beta\alpha$. Let $S = (\mathbb{N} \cup \{0\}) \times (\mathbb{N} \cup \{0\})$ (resp. $\mathbb{Z} \times \mathbb{Z}$) endowed with the lexicographic order, or the reverse lexicographic order, or the product order of the usual order of $\mathbb{N} \cup \{0\}$ (resp. $\mathbb{Z}$), and define $\lambda : S \longrightarrow R\,\mathrm{End}(R)$ via $\lambda(m,n) = \alpha^m\beta^n$ for any $m, n \in \mathbb{N} \cup \{0\}$ (resp. $m, n \in \mathbb{Z}$). Then $[[R^{S,\leq}, \lambda]] = R[[x, y; \alpha, \beta]]$ (resp. $R[[x, y, x^{-1}, y^{-1}; \alpha, \beta]]$), in which $(ax^m y^n)(bx^p y^q) = a\alpha^m\beta^n(b)x^{m+p}y^{n+q}$ for any $m, n, p, q \in \mathbb{N} \cup \{0\}$ (resp. $m, n, p, q \in \mathbb{Z}$).

3 Triangulating Dimensions

Let α be a ring endomorphism of R. According to [13] or [14], α is called a *rigid* endomorphism if $r\alpha(r) = 0$ implies $r = 0$ for $r \in R$. A ring R is said to be *α-rigid* if there exists a rigid endomorphism α of R. Clearly any rigid endomorphism is a monomorphism and any α-rigid ring is reduced. Generalizing these concepts, we give the following definition (see [15]).

Definition 3.1 *Let α be a ring endomorphism of R. Then α is called a weakly rigid endomorphism if*

(1) *α is a monomorphism, and*

(2) *if $a, b \in R$ are such that $ab = 0$ then $a\alpha(b) = \alpha(a)b = 0$.*

Clearly, the identity map of R is weakly rigid. Every monomorphism of rings without non-zero zero-divisors is weakly rigid.

Let α be a rigid endomorphism of R. It was shown in [13] that if $ab = 0$ then $a\alpha^n(b) = \alpha^n(a)b = 0$ for any positive integer n. Thus any rigid endomorphism is weakly rigid. But the converse is not true. For example, supposing that the ring R is not reduced, then the identity map of R is weakly rigid but not rigid. In fact, if α is a ring endomorphism of R, then, by [14, Proposition 3], α is rigid if and only if α is weakly rigid and R is reduced. Further examples of weakly rigid endomorphisms of rings can be found in [15].

We say $\lambda : S \longrightarrow R\,\mathrm{End}(R)$ is *weakly rigid* if for every $s \in S$, $\lambda(s)$ is a weakly rigid endomorphism of R. For example, if α is a weakly rigid endomorphism of R, then $\lambda : \mathbb{N}\cup\{0\} \longrightarrow R\,\mathrm{End}(R)$: $\lambda(0) = 1$, $\lambda(k) = \alpha^k$ for any $k \in \mathbb{N}$ is weakly rigid.

Lemma 3.2 *Let λ be weakly rigid. Then*

(1) *$\lambda(0) = 1$, and*

(2) *for any $s \in S$ and any $b^2 = b \in R$, $\lambda(s)(b) = b$.*

Proof Clearly $\lambda(0) = \lambda(0)\lambda(0)$. Thus $\lambda(0) = 1$ since $\lambda(0)$ is a monomorphism.

Let $b^2 = b \in R$. Then $b(1-b) = 0$. Thus $\lambda(s)(b)(1-b) = 0$ since $\lambda(s)$ is weakly rigid. Hence $\lambda(s)(b) = \lambda(s)(b)b$. On the other hand, $\lambda(s)(b) = \lambda(s)(b)\lambda(s)(b)$, thus $(1 - \lambda(s)(b))\lambda(s)(b) = 0$. So $\lambda(s)(1 - \lambda(s)(b))\lambda(s)(b) = 0$ since $\lambda(s)$ is weakly rigid, which implies that $(1 - \lambda(s)(b))b = 0$ since $\lambda(s)$ is a monomorphism. Hence $b = \lambda(s)(b)b$. Now it follows that $\lambda(s)(b) = b$.

We shall henceforth assume that $\lambda : S \longrightarrow R\,\mathrm{End}(R)$ is weakly rigid and satisfies the condition that $\lambda(u + v) = \lambda(u)\lambda(v)$ for any $u, v \in S$.

Recall from [3, 5, 6] that an ordered set $\{b_1, \ldots, b_n\}$ of nonzero distinct idempotents in a ring R is called a set of *left triangulating idempotents* of R if all the following hold:

(i) $1 = b_1 + \cdots + b_n$;

(ii) $b_1 \in \mathscr{S}_l(R)$; and

(iii) $b_{k+1} \in \mathscr{S}_l(a_k R a_k)$, where $a_k = 1 - (b_1 + \cdots + b_k)$, for $1 \leq k \leq n - 1$.

Similarly we can define a set of right triangulating idempotents of R using (i), $b_1 \in \mathscr{S}_r(R)$, and $b_{k+1} \in \mathscr{S}_r(a_k R a_k)$.

Let B be a set of left triangulating idempotents of R and Γ a ring extension of R. From [7], we say Γ is *B-triangularly linked to R* if whenever $b \in B$ and $0 \neq a \in \mathscr{S}_l(b\Gamma b)$, then there exists $0 \neq a_0 \in \mathscr{S}_l(bRb)$ such that $a_0\Gamma \subseteq a\Gamma$. We say Γ is *B-triangularly compatible with R* if B is a set of left triangulating idempotents of Γ.

Let $r \in R$. Define a mapping $c_r \in [[R^{S,\leq}, \lambda]]$ as follows:

$$c_r(0) = r, \quad c_r(s) = 0, \quad 0 \neq s \in S.$$

Then $\{c_r | r \in R\}$ is isomorphic to a subring of $[[R^{S,\leq}, \lambda]]$ by Lemma 3.2.

Lemma 3.3 *Let $(S, \leq)$ be a strictly ordered monoid. If $B = \{b_1, \ldots, b_n\}$ is a set of left triangulating idempotents of R, then $\{c_{b_1}, \ldots, c_{b_n}\}$ is a set of left triangulating idempotents of $[[R^{S,\leq}, \lambda]]$.*

Proof From Lemma 3.2, it is easy to see that c_b is an idempotent of $[[R^{S,\leq}, \lambda]]$ for any $b \in B$. Since $1 = b_1 + \cdots + b_n$, it is easy to see that $c_1 = c_{b_1} + \cdots + c_{b_n}$. For any $s \in S$ and any $r \in R$, $(r - b_1 r)\lambda(s)(b_1) = 0$ since $(r - b_1 r)b_1 = 0$ and $\lambda(s)$ is weakly rigid. Thus for any $f \in [[R^{S,\leq}, \lambda]]$ and any $s \in S$, $(c_{b_1} f c_{b_1})(s) = \sum_{(u,v,w)\in X_s(c_{b_1}, f, c_{b_1})} c_{b_1}(u)\lambda(u)(f(v))\lambda(u + v)(c_{b_1}(w)) = b_1\lambda(0)(f(s))\lambda(s)(b_1) = b_1 f(s)\lambda(s)(b_1) = f(s)\lambda(s)(b_1) = \sum_{(u,v)\in X_s(f,c_{b_1})} f(u)\lambda(u) (c_{b_1}(v)) = (f c_{b_1})(s)$. Thus $c_{b_1} f c_{b_1} = f c_{b_1}$. This means that $c_{b_1} \in \mathscr{S}_l([[R^{S,\leq}, \lambda]])$. Note $b_{k+1} \in \mathscr{S}_l(a_k R a_k)$, where $a_k = 1 - (b_1 + \cdots + b_k)$, for $1 \leq k \leq n - 1$. Thus $(a_k r - b_{k+1} a_k r)a_k b_{k+1} = 0$, and so $(a_k r - b_{k+1} a_k r)\lambda(s)(a_k b_{k+1}) = 0$ since $\lambda(s)$ is weakly rigid. It is easy to see that $c_{a_k} = c_1 - (c_{b_1} + \cdots + c_{b_k})$. Thus for any $f \in [[R^{S,\leq}, \lambda]]$ and any $s \in S$, $(c_{b_{k+1}} c_{a_k} f c_{a_k} c_{b_{k+1}})(s) = \sum_{(u,v,w,p,q)} c_{b_{k+1}}(u)\lambda(u)(c_{a_k}(v))\lambda(u+v)(f(w))\lambda(u+v+w)(c_{a_k}(p))\lambda(u+v+w+p)(c_{b_{k+1}}(q)) = b_{k+1} a_k f(s)\lambda(s)(a_k b_{k+1}) = a_k f(s)\lambda(s)(a_k b_{k+1}) = (c_{a_k} f c_{a_k} c_{b_{k+1}})(s)$. So $c_{b_{k+1}} c_{a_k} f c_{a_k} c_{b_{k+1}} = c_{a_k} f c_{a_k} c_{b_{k+1}}$. This means that $c_{b_{k+1}} \in \mathscr{S}_l(c_{a_k}[[R^{S,\leq}, \lambda]]c_{a_k})$. Now the result follows.

Recall that a monoid S is called *torsion-free* if the following property holds: If $s, t \in S$, if k is an integer, $k \geq 1$ and $ks = kt$, then $s = t$. If $(S, \leq)$ is a strictly totally ordered monoid, then by [1, 3.2], S is cancellative and torsion-free. It was proved in [16, Lemma 3] that if $(S, \leq)$ is a strictly totally ordered monoid satisfying that $0 \leq s$ for all $s \in S$ and if ϕ is a left semicentral idempotent of $[[R^{S,\leq}]]$, then $\phi(0) \in R$ is a left semicentral idempotent and $\phi[[R^{S,\leq}]] = c_{\phi(0)}[[R^{S,\leq}]]$. Here we have

Lemma 3.4 *Let R be a ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S satisfying that $0 \leq s$ for all $s \in S$. If $\phi \in [[R^{S,\leq}, \lambda]]$ is a left semicentral idempotent, then $\phi(0) \in R$ is a left semicentral idempotent and $\phi[[R^{S,\leq}, \lambda]] = c_{\phi(0)}[[R^{S,\leq}, \lambda]]$.*

Proof For any $r \in R$, $c_r\phi = \phi c_r \phi$. Thus by Lemma 3.2,

$$r\phi(0) = r\lambda(0)(\phi(0)) = (c_r\phi)(0) = (\phi c_r \phi)(0) = \phi(0)\lambda(0)(r\phi(0)) = \phi(0)r\phi(0),$$

which implies that $\phi(0)$ is a left semicentral idempotent of R.

If $\phi(0) = 0$, then $\phi = 0$. Otherwise, suppose that $\phi \neq 0$. Then $\mathrm{supp}(\phi) \neq \varnothing$. By [1], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all $s, t \in S$, $s \leq t$ implies $s \leq' t$). Since $\mathrm{supp}(\phi)$ is a non-empty artinian and narrow subset of S, the set $\mathrm{Min}(\mathrm{supp}(\phi))$ of minimal elements of $\mathrm{supp}(\phi)$ is finite and non-empty. Let $\mathrm{Min}(\mathrm{supp}(\phi)) = \{s_1, s_2, \ldots, s_n\}$ with $s_1 <' s_i$ for every $i = 2, \ldots, n$. Then $0 \neq \phi(s_1) = \phi^2(s_1) = \sum_{(u,v)\in X_{s_1}(\phi,\phi)} \phi(u)\lambda(u)(\phi(v)) = 0$, a contradiction. This shows that $\phi = 0$. Thus $\phi[[R^{S,\leq}, \lambda]] = c_{\phi(0)}[[R^{S,\leq}, \lambda]]$. Now suppose that $\phi(0) \neq 0$. If $\mathrm{supp}(\phi) = \{0\}$, then clearly $\phi = c_{\phi(0)}$. So suppose that $\mathrm{supp}(\phi) \neq \{0\}$.

Denote the minimal element under the order $\leq'$ of $\mathrm{supp}(\phi) - \{0\}$ by t. Then $r\phi(t) = r\lambda(0)(\phi(t)) = (c_r\phi)(t) = (\phi c_r \phi)(t) = \sum_{(u,v)\in X_t(\phi,\phi)} \phi(u)\lambda(u)(r\phi(v)) = \phi(t)\lambda(t)(r\phi(0)) + \phi(0)\lambda(0)(r\phi(t)) = \phi(t)\lambda(t)(r\phi(0)) + \phi(0)r\phi(t)$ since $\phi(s) = 0$ for any $s \in S$ with $0 <' s <' t$. Multiply the right-hand side by $\phi(0)$ to get $r\phi(t)\phi(0) = \phi(t)\lambda(t)(r\phi(0))\phi(0) + \phi(0)r\phi(t)\phi(0)$. But

$\phi(0)r\phi(t)\phi(0) = r\phi(t)\phi(0)$. Hence $\phi(t)\lambda(t)(r\phi(0))\phi(0) = 0$, and $r\phi(t) = \phi(0)r\phi(t)$. From the weak rigidness of $\lambda(t)$, it follows that $\lambda(t)(\phi(t))\lambda(t)(r\phi(0))\lambda(t)(\phi(0)) = 0$. Thus $\phi(t)r\phi(0) = 0$ by the weak rigidness of $\lambda(t)$ again.

By analogy with the proof of [16, Lemma 3], bearing in mind that λ is weakly rigid, we get that for any $0 <' w \in \mathrm{supp}(\phi)$,

$$r\phi(w) = \phi(0)r\phi(w), \quad \phi(w)r\phi(0) = 0, \quad \forall\, r \in R.$$

Thus for any $0 <' w \in \mathrm{supp}(\phi)$, $\phi(w)\lambda(w)(\phi(0)) = 0$ since $\lambda(w)$ is weakly rigid. Now it is easy to see that $c_{\phi(0)} = \phi c_{\phi(0)}$ and $\phi = c_{\phi(0)}\phi$, which imply that $\phi[[R^{S,\leq}, \lambda]] = c_{\phi(0)}[[R^{S,\leq}, \lambda]]$.

Lemma 3.5 *Let R be a ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S satisfying that $0 \leq s$ for all $s \in S$. If B is a set of left triangulating idempotents of R, then $[[R^{S,\leq}, \lambda]]$ is B-triangularly linked to R.*

Proof Suppose that $b \in B$ and $0 \neq \phi \in \mathscr{S}_l(c_b[[R^{S,\leq}, \lambda]]c_b)$. For every $f \in [[R^{S,\leq}, \lambda]]$, it is easy to see that $(c_bfc_b)(s) = bf(s)\lambda(s)(b) = bf(s)b \in bRb$ for any $s \in S$ by Lemma 3.2. Thus $c_bfc_b \in [[(bRb)^{S,\leq}, \lambda]]$. Conversely if $g \in [[(bRb)^{S,\leq}, \lambda]]$, then clearly $g = c_bgc_b \in c_b[[R^{S,\leq}, \lambda]]c_b$. Thus $c_b[[R^{S,\leq}, \lambda]]c_b = [[(bRb)^{S,\leq}, \lambda]]$. Now from Lemma 3.4, there exists $0 \neq a \in \mathscr{S}_l(bRb)$ such that $c_a[[(bRb)^{S,\leq}, \lambda]] \subseteq \phi[[(bRb)^{S,\leq}, \lambda]]$. Thus $c_a = \phi c_a$ and so $c_a[[R^{S,\leq}, \lambda]] \subseteq \phi[[R^{S,\leq}, \lambda]]$.

Lemma 3.6 *Let S be cancellative and torsion-free and R a reduced ring. If $f, g \in [[R^{S,\leq}, \lambda]]$ are such that $fg = 0$, then $f(u)g(v) = 0$ for any $u \in \mathrm{supp}(f)$ and any $v \in \mathrm{supp}(g)$.*

Proof By analogy with the proof of [17, Lemma 3.1], we can complete the proof.

Lemma 3.7 *Let S be cancellative and torsion-free and R a reduced ring. If $\phi \in [[R^{S,\leq}, \lambda]]$ is an idempotent, then there exists an idempotent $e \in R$ such that $\phi = c_e$.*

Proof From $\phi(c_1 - \phi) = 0$ it follows that $\phi(u)(c_1 - \phi)(v) = 0$ for any $u \in \mathrm{supp}(\phi)$ and $v \in \mathrm{supp}(c_1 - \phi)$. If $0 \neq s \in \mathrm{supp}(\phi)$, then $\phi(s)^2 = 0$, and so $\phi(s) = 0$ since R is reduced, a contradiction. Thus $\mathrm{supp}(\phi) \subseteq \{0\}$. Hence $\phi = c_{\phi(0)}$. From $\phi(0)(1 - \phi(0)) = 0$ it follows that $\phi(0) \in R$ is an idempotent.

The following lemma shows that if R is reduced, then the condition that $0 \leq s$ for all $s \in S$ in Lemma 3.5 can be omitted.

Lemma 3.8 *Let R be a reduced ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S. If B is a set of left triangulating idempotents of R, then $[[R^{S,\leq}, \lambda]]$ is B-triangularly linked to R.*

Proof Note that every reduced ring is Abelian. The result follows by analogy with the proof of Lemma 3.5, and by using Lemma 3.7.

A set $\{b_1, \ldots, b_n\}$ of left (right) triangulating idempotents is said to be complete if each b_i is also semicentral reduced. Note that any complete set of primitive idempotents determines a complete set of left triangulating idempotents [5, Proposition 2.18].

Lemma 3.9 ([5, Proposition 1.3]) *R has a* (*respectively, complete*) *set of left triangulating idempotents if and only if R has a* (*respectively, complete*) *generalized triangular matrix representation.*

From [5] the number of elements in a complete set of left triangulating idempotents is unique for a given ring R (which has such a set) and this is also the number of elements in any complete set of right triangulating idempotents of R. Thus it is natural to see that R has *triangulating dimension* n, written as $\mathrm{Tdim}(R) = n$, if R has a complete set of left triangulating idempotents with exactly n elements. If R has no complete set of left triangulating idempotents, then we say R has infinite triangulating dimension, denoted as $\mathrm{Tdim}(R) = \infty$. Note that R is semicentral reduced if and only if $\mathrm{Tdim}(R) = 1$.

Lemma 3.10 ([7, Proposition 4.3]) *Let Γ be a ring extension of R. If Γ is B-triangularly linked to R and B-triangularly compatible with R for every set B of left triangulating idempo-*

tents of R, then $\mathrm{Tdim}(R) = \mathrm{Tdim}(\Gamma)$.

Theorem 3.11 *Let R be a ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S. If one of the following conditions holds, then $[[R^{S,\leq}, \lambda]]$ has the same triangulating dimension as R:*

(1) *$0 \leq s$ for all $s \in S$.*

(2) *R is reduced.*

Proof This follows from Lemmas 3.3, 3.5, 3.8, 3.10.

If M is a right R-module, we let $[M^{S,\leq}]$ be the set of all maps $\phi : S \longrightarrow M$ such that the set $\mathrm{supp}(\phi) = \{s \in S | \phi(s) \neq 0\}$ is finite. Now $[M^{S,\leq}]$ can be turned into a right $[[R^{S,\leq}]]$-module under some additional conditions. The addition in $[M^{S,\leq}]$ is componentwise and the scalar multiplication is defined as follows

$$(\phi f)(s) = \sum_{t \in S} \phi(s+t) f(t), \qquad \text{for every} \qquad s \in S,$$

where $f \in [[R^{S,\leq}]]$, and $\phi \in [M^{S,\leq}]$. Since the set $\mathrm{supp}(\phi)$ is finite, this multiplication is well-defined. If $(S, \leq)$ is a strictly totally ordered monoid which is also artinian, then, from [18], $[M^{S,\leq}]$ becomes a right $[[R^{S,\leq}]]$-module, which is called the *generalized Macaulay–Northcott module*. For example, if $S = \mathbb{N} \cup \{0\}$ and $\leq$ is the usual order, then $[M^{\mathbb{N}\cup\{0\},\leq}] \cong M[x^{-1}]$, the usual right $R[[x]]$-module which is called the Macaulay–Northcott module in [19]. Further work on generalized Macaulay–Northcott modules appears in [20, 21].

The following result appears in [6, Corollary 1.11].

Lemma 3.12 *Let M be a right R-module and $E = \mathrm{End}_R(M)$. The following conditions are equivalent*:

(i) *E has a complete generalized triangular matrix representation*;

(ii) *M has ACC and DCC on fully invariant direct summands*;

(iii) *M has only finitely many distinct fully invariant direct summands.*

Corollary 3.13 *Let S be a cancellative and torsion-free monoid and $\leq$ a strict order on S which is also artinian. If the right R-module M has only finitely many distinct fully invariant direct summands, then the ring $\mathrm{End}_{[[R^{S,\leq}]]}([M^{S,\leq}])$ has a complete generalized triangular matrix representation.*

Proof It follows from [21, Theorem 2.1] that there exists an isomorphism of rings

$$\mathrm{End}_{[[R^{S,\leq}]]}([M^{S,\leq}]) \cong [[\mathrm{End}_R(M)^{S,\leq}]].$$

If $s < 0$, then $\cdots < 3s < 2s < s < 0$, a contradiction. Thus $(S, \leq)$ satisfies the condition that $0 \leq s$ for every $s \in S$. Now the result follows from Theorem 3.11 and Lemma 3.12.

4 PWP Rings

Recall that R is (*quasi-*) *Baer* if the right annihilator of every nonempty subset (every right ideal) of R is generated by an idempotent. Clark defined quasi-Baer rings in [22] and used them to characterize when a finite dimensional algebra with unity over an algebraically closed field is isomorphic to a twisted matrix units semigroup algebra. Every prime ring is quasi-Baer. In [23] Pollingher and Zaks showed that the class of quasi-Baer rings is closed under $n \times n$ matrix rings and under $n \times n$ upper (or lower) triangular matrix rings. Birkenmeier, Kim and Park proved in [4, Theorem 1.8] that a ring R is quasi-Baer if and only if $R[X]$ is quasi-Baer if and only if $R[[X]]$ is quasi-Baer, where X is an arbitrary nonempty set of not necessarily commuting indeterminates. Hong, Kim and Kwak showed in [13, Corollary 22] that if α is a rigid endomorphism of R, then R is a quasi-Baer ring if and only if $R[[x; \alpha]]$ is a quasi-Baer ring. Further work on quasi-Baer rings appears in [7, 13, 15, 16].

Lemma 4.1 *Let R be a ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S. If R is quasi-Baer, then $[[R^{S,\leq}, \lambda]]$ is quasi-Baer.*

Proof Let Q be a right ideal of $[[R^{S,\leq},\lambda]]$. Since S is cancellative and torsion-free, by [1], there exists a compatible strict total order $\leq'$ on S, which is finer than $\leq$ (that is, for all $s,t\in S$, $s\leq t$ implies $s\leq' t$). For every $0\neq f\in[[R^{S,\leq},\lambda]]$, since $\mathrm{supp}(f)$ is a non-empty artinian and narrow subset of S, the set $\mathrm{Min}(\mathrm{supp}(f))$ of minimal elements of $\mathrm{supp}(f)$ is finite and non-empty. Thus there exists a minimal element of $\mathrm{supp}(f)$ under the total order $\leq'$, which will be denoted by $\pi'(f)$.

For every $s\in S$, set $I_s=\{f(s)|f\in Q,\pi'(f)=s\}$, and $I=\cup_{s\in S}I_s$. Let J be the right ideal of R generated by I. Then there exists an idempotent e of R such that $r_R(J)=eR$. We will show that $r_{[[R^{S,\leq},\lambda]]}(Q)=c_e[[R^{S,\leq},\lambda]]$.

Suppose that $f\in Q$. Then $fc_e\in Q$. If $fc_e\neq 0$, then set $\pi'(fc_e)=t$. Then $(fc_e)(t)\neq 0$. On the other hand, $f(t)\lambda(t)(e)=\sum_{(u,v)\in X_t(f,c_e)}f(u)\lambda(u)(c_e(v))=(fc_e)(t)\in I_t\subseteq J$, thus $f(t)\lambda(t)(e)e=0$. Hence, by Lemma 3.2, $f(t)\lambda(t)(e)=0$, a contradiction. Hence $fc_e=0$. This means that $c_e[[R^{S,\leq},\lambda]]\subseteq r_{[[R^{S,\leq},\lambda]]}(Q)$.

Suppose that $g\in r_{[[R^{S,\leq},\lambda]]}(Q)$ and $g\neq 0$. Set $\pi'(g)=s$. For any $a\in J$, there exist $s_1,\dots,s_n\in S,f_1,\cdots,f_n\in Q$, and $r_1,\dots,r_n\in R$, such that $a=f_1(s_1)r_1+\cdots+f_n(s_n)r_n$, and $\pi'(f_i)=s_i, f_i(s_i)\in I_{s_i}$, $i=1,\dots,n$. Clearly we can assume that $f_i(s_i)r_i\neq 0, i=1,\dots,n$. Thus $(f_ic_{r_i})(s_i)=f_i(s_i)\lambda(s_i)(r_i)\neq 0$ since $\lambda(s_i)$ is weakly rigid. For any $t<'s_i$, if $(f_ic_{r_i})(t)=f_i(t)\lambda(t)(r_i)\neq 0$, then $f_i(t)r_i\neq 0$ since $\lambda(t)$ is weakly rigid. Thus $f_i(t)\neq 0$, a contradiction with $\pi'(f_i)=s_i$. Hence $\pi'(f_ic_{r_i})=s_i$. Since $f_ic_{r_i}\in Q$, we have $(f_ic_{r_i})g=0$. Thus
$$0=(f_ic_{r_i}g)(s_i+s)=\sum_{(u,v)\in X_{s_i+s}(f_ic_{r_i},g)}(f_ic_{r_i})(u)\lambda(u)(g(v)).$$
Since s_i and s are the minimal elements of $\mathrm{supp}(f_ic_{r_i})$ and $\mathrm{supp}(g)$, respectively, under the total order $\leq'$, if $u\in\mathrm{supp}(f_ic_{r_i})$ and $v\in\mathrm{supp}(g)$ are such that $u+v=s_i+s$, then $s_i\leq' u$ and $s\leq' v$. If $s_i<'u$ then $s_i+s<'u+v=s_i+s$, a contradiction. Thus $u=s_i$. Similarly, $v=s$. Hence
$$0=\sum_{(u,v)\in X_{s_i+s}(f_ic_{r_i},g)}(f_ic_{r_i})(u)\lambda(u)(g(v))=(f_ic_{r_i})(s_i)\lambda(s_i)(g(s)).$$
So $\lambda(s_i)((f_ic_{r_i})(s_i)(g(s)))=0$. Thus $(f_ic_{r_i})(s_i)g(s)=0$. Hence $ag(s)=0$. This implies that $g(s)\in r_R(J)=eR$. Therefore $g(s)=eg(s)$.

We claim that for any $u\in\mathrm{supp}(g)$, $g(u)=eg(u)$.

Suppose that $u\in\mathrm{supp}(g)$. Assume that $g(v)=eg(v)$ for any $v\in\mathrm{supp}(g)$ with $v<'u$. We will show that $g(u)=eg(u)$. Denote
$$g_u(x)=\begin{cases}g(x), & x<'u,\\ 0, & u\leq' x.\end{cases}$$
Then $\pi'(g-g_u)=u$. By hypothesis it is easy to see that $g_u=c_eg_u\in c_e[[R^{S,\leq},\lambda]]\leq r_{[[R^{S,\leq},\lambda]]}(Q)$. Thus $g-g_u\in r_{[[R^{S,\leq},\lambda]]}(Q)$. By analogy with the proof above, it follows that $(g-g_u)(u)=e(g-g_u)(u)$, which implies that $g(u)=eg(u)$. Thus our claim holds.

Now from $(c_eg)(t)=\sum_{(u,v)\in X_t(c_e,g)}c_e(u)\lambda(u)(g(v))=c_e(0)\lambda(0)(g(t))=eg(t)=g(t)$ it follows that $g=c_eg\in c_e[[R^{S,\leq},\lambda]]$. Therefore $r_{[[R^{S,\leq},\lambda]]}(Q)=c_e[[R^{S,\leq},\lambda]]$, and so $[[R^{S,\leq},\lambda]]$ is quasi-Baer.

Theorem 4.2 *Let R be a PWP ring and S a cancellative and torsion-free monoid and $\leq$ a strict order on S. If one of the following conditions holds, then $[[R^{S,\leq},\lambda]]$ is a PWP ring*:

(i) $0\leq s$ *for all* $s\in S$.

(ii) R *is reduced.*

Proof This follows from Lemma 4.1 and Theorem 3.11.

Note that there exists a PWP ring which is not reduced. For example, the ring $\begin{pmatrix}\mathbb{Z} & \mathbb{Z}\\ 0 & \mathbb{Z}\end{pmatrix}$, where $\mathbb{Z}$ is the ring of integers, is a PWP ring, but not reduced. Note that a PWP reduced ring is a finite direct sum of domains.

Corollary 4.3 *Let $(S, \leq)$ be a strictly totally ordered monoid and R a PWP ring. If either $0 \leq s$ for all $s \in S$, or R is reduced, then $[[R^{S,\leq}, \lambda]]$ is a PWP ring.*

Proof If $\leq$ is a strict total order on S, then, by [2, 3.2], S is cancellative and torsion-free. Thus the result follows from Theorem 4.2.

The following corollaries will give other examples of PWP rings.

Corollary 4.4 *Let $(S_1, \leq_1), \ldots, (S_n, \leq_n)$ be strictly totally ordered monoids and R a PWP ring. Denote by $(lex \leq)$ and $(revlex \leq)$ the lexicographic order, the reverse lexicographic order, respectively, on the monoid $S_1 \times \cdots \times S_n$. If either every $(S_i, \leq_i)$ satisfies the condition that $0 \leq_i s$ for every $s \in S_i$, $i = 1, \ldots, n$, or R is reduced, then $[[R^{S_1 \times \cdots \times S_n, (lex \leq)}, \lambda]]$ and $[[R^{S_1 \times \cdots \times S_n, (revlex \leq)}, \lambda]]$ are PWP rings.*

Proof It is easy to see that $(S_1 \times \cdots \times S_n, (lex \leq))$ is a strictly totally ordered monoid which also satisfies the following condition:

$$0 (lex \leq)(s_1, \ldots, s_n), \quad \forall (s_1, \ldots, s_n) \in (S_1 \times \cdots \times S_n, (lex \leq))$$

if every $(S_i, \leq_i)$ satisfies the condition that $0 \leq_i s$ for every $s \in S_i$, $i = 1, \ldots, n$. Thus by Corollary 4.3, $[[R^{S_1 \times \cdots \times S_n, (lex \leq)}, \lambda]]$ is a PWP ring.

The proof for ring $[[R^{S_1 \times \cdots \times S_n, (revlex \leq)}, \lambda]]$ is similar.

Corollary 4.5 *Let R be a PWP ring. Then so is the ring $[[R^{\mathbb{N}_{\geq 1}, \leq}]]$.*

Any submonoid of the additive monoid $\mathbb{N} \cup \{0\}$ is called a *numerical monoid*. We have

Corollary 4.6 *Let S be a numerical monoid and $\leq$ the usual natural order of $\mathbb{N} \cup \{0\}$. Then $[[R^{S,\leq}]]$ has the same triangulating dimension as R. Furthermore if R is a PWP ring, then so is $[[R^{S,\leq}]]$.*

Corollary 4.7 *Let S be a submonoid of $(\mathbb{N} \cup \{0\})^n$ $(n \geq 2)$, endowed with the order $\leq$ induced by the product order, or lexicographic order or reverse lexicographic order. If R is a PWP ring, then so is $[[R^{S,\leq}]]$.*

Proof Since S is a torsion-free and cancellative monoid, the result follows from Theorem 4.2.

Corollary 4.8 *Suppose that $(S, \leq)$ is a strictly totally ordered monoid which is also artinian. Let M be a right R-module and $E = \mathrm{End}_R(M)$. If E is a PWP ring, then so is $\mathrm{End}_{[[R^{S,\leq}]]}([M^{S,\leq}])$.*

Proof This follows from Theorem 4.2 and from the proof of Corollary 3.13.

Note that the maps λ in Examples 2.2(6) and 2.2(7) are weakly rigid. Thus the quantum plane $R[x][y;\alpha]$ in Example 2.2(6) and the universal enveloping algebra U of $(V, [,])$ in Example 2.2(7) are PWP rings.

Let R be a PWP ring with a complete set of left triangulating idempotents $B = \{b_1, \ldots, b_n\}$. In [7, Theorem 4.8], it was shown that if α is a ring automorphism such that $\alpha(bR) \subseteq bR$ for all $b \in B$, then $R[x;\alpha]$ and $R[[x;\alpha]]$ are PWP rings. Here we have

Corollary 4.9 *Let R be a PWP ring and α a weakly rigid endomorphism of R. Then $R[x;\alpha]$ and $R[[x;\alpha]]$ are PWP rings.*

In [7, Theorem 4.8], it was shown that if R is a PWP ring, then $R[x, x^{-1}]$ and $R[[x, x^{-1}]]$ are PWP rings. Here we have

Corollary 4.10 *Let R be a reduced PWP ring and α a weakly rigid automorphism of R. Then $R[x, x^{-1};\alpha]$ and $R[[x, x^{-1};\alpha]]$ are PWP rings.*

Corollary 4.11 *Let R be a reduced PWP ring and α and β be weakly rigid ring endomorphisms (respectively, ring automorphisms) of R such that $\alpha\beta = \beta\alpha$. Then $R[[x, y;\alpha, \beta]]$ (resp. $R[[x, y, x^{-1}, y^{-1};\alpha, \beta]]$) is a PWP ring.*

Remark 4.12 Let $(S, \leq)$ be a strictly totally ordered monoid which is also artinian. Then the set $X_s = \{(u, v) | u + v = s, u, v \in S\}$ is finite for any $s \in S$. Let V be a free Abelian additive group with the base consisting of elements of S. It was noted in [21] that V is a coalgebra over $\mathbb{Z}$ with the comultiplication map and the counit map as follows:

$$\Delta(s) = \sum_{(u,v)\in X_s} u \otimes v, \qquad \epsilon(s) = \begin{cases} 1, & s = 0, \\ 0, & s \neq 0, \end{cases}$$

and $[[R^{S,\leq}]] \cong \mathrm{Hom}(V, R)$, the dual algebra with multiplication

$$f * g = (f \otimes g)\Delta \qquad \forall\, f, g \in \mathrm{Hom}(V, R).$$

If R is a PWP ring, then from Theorem 4.2 we know that the ring $\mathrm{Hom}(V, R)$ is a PWP ring.

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