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Prime Submodules and Flat Modules

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Abstract In this paper, some characterizations of prime submodules in flat modules and, particularly, in free modules are given. Furthermore, the height of prime submodules and some saturated chain of prime submodules are also given.

Keywords flat and absolutely flat modules, generalized principal ideal theorem, prime submodules, rank of modules
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1 Introduction

Throughout this paper all rings are considered to be commutative with identity and all modules are unitary.

Let M be an R-module. A proper submodule N of M is said to be a prime submodule of M, if the condition $ra \in N$, $r \in R$ and $a \in M$. implies that $a \in N$ or $rM \subseteq N$.

Recall that $(N : M) = \{r \in R | \ rM \subseteq N\}$. If N is a prime submodule of M and $P = (N : M)$, we say that N is a P-prime submodule of M. It is easy to see that P is a prime ideal of R (see, for example, [1], or [2]).

A prime submodule N of a module M is called a minimal prime submodule over a submodule K of M, if $K \subseteq N$ and there does not exist a prime submodule L of M such that $K \subseteq L \subset N$.

If n is a non-negative integer, and for each integer i, $0 \leq i \leq n$, N_i is a prime (or P-prime) submodule of M, then the chain

$$N_n \subset \cdots \subset N_2 \subset N_1 \subset N_0$$

is called a chain of prime (P-prime) submodules of length n. Also if N is a prime submodule of M then by the height (or P-height) of N, which is denoted by ht N, (or ht_P N,) we mean the height of a chain of prime (P-prime) submodules of M, such as above, for which $N_0 = N$ such that there does not exist a chain of greater length with this property.

It is said that the principal ideal theorem (PIT) holds for M, if for every prime submodule N of M minimal over a cyclic submodule, ht $N \leq 1$.

We say that the generalized principal ideal theorem (GPIT) holds for M, if for every positive integer n and every prime submodule N of M minimal over a submodule generated by n elements, ht $N \leq n$.

2 Some Remarks on Prime Submodules

Lemma 2.1 *Let M be an R-module, N a submodule of M and S a multiplicatively closed subset of R.*

i) *If N is a P-prime submodule of M such that $P \cap S = \emptyset$, then $S^{-1}N$ is an $S^{-1}P$-prime submodule of $S^{-1}M$ as an $S^{-1}R$-module, and $(S^{-1}N) \cap M = N$.*

ii) *If T is a Q-prime submodule of $S^{-1}M$ as an $S^{-1}R$-module, then $T \cap M$ is a $Q \cap R$-prime submodule of M, $S^{-1}(T \cap M) = T$, and $(Q \cap R) \cap S = \emptyset$.*

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Proof See [3, Proposition 1].

Corollary 2.2 *If N is a prime submodule of an R-module M such that $(N : M) \cap S = \emptyset$, then* $\mathrm{ht}\, N = \mathrm{ht}\, S^{-1}N$.

Proof The proof is straightforward by Lemma 2.1.

Lemma 2.3 *A proper submodule N of an R-module M is a prime submodule of M if and only if for every $r \in R$, the natural homomorphism $f_r : \frac{M}{N} \to \frac{M}{N}$, $f_r(m+N) = rm+N$, is one-to-one or zero.*

Proof The proof is easy (see [4, p. 273]).

Theorem 2.4 *Let F be a flat R-module and N a prime submodule of an R-module M. If $F \otimes N$ is a proper submodule of $F \otimes M$, then $F \otimes N$ is a prime submodule of $F \otimes M$.*

Proof Since F is a flat R-module, then $F \otimes \frac{M}{N} \cong \frac{F\otimes M}{F\otimes N}$. Let $r \in R$ and the natural homomorphism $f_r : \frac{M}{N} \to \frac{M}{N}$, $f_r(m+N) = rm+N$, be zero. We, obviously, observe that the natural homomorphism $g_r : F \otimes \frac{M}{N} \to F \otimes \frac{M}{N}$ defined by $g_r(f \otimes (m+N)) = f \otimes (rm+N)$, is zero. Thus, the natural homomorphism $h_r : \frac{F\otimes M}{F\otimes N} \to \frac{F\otimes M}{F\otimes N}$, given by $h_r((f\otimes m)+F\otimes N) = r(f\otimes m)+F\otimes N$, is zero.

When the homomorphism $f_r : \frac{M}{N} \to \frac{M}{N}$ is a monomorphism, then since F is a flat R-module, the homomorphism $g_r : F \otimes \frac{M}{N} \to F \otimes \frac{M}{N}$ is a monomorphism. Thus the homomorphism $h_r : \frac{F\otimes M}{F\otimes N} \to \frac{F\otimes M}{F\otimes N}$ is a monomorphism. Hence by Lemma 2.3, $F \otimes N$ is a prime submodule of $F \otimes M$.

Proposition 2.5 *Let F be a faithfully flat R-module and N a submodule of an R-module M. Then N is a prime submodule of M, if and only if $F \otimes N$ is a prime submodule of $F \otimes M$.*

Proof Let N be a prime submodule of M and $F \otimes N = F \otimes M$. Therefore, $0 \longrightarrow F \otimes N \longrightarrow F \otimes M \longrightarrow 0$ is an exact sequence, and since F is a faithfully flat R-module, then $0 \longrightarrow N \longrightarrow M \longrightarrow 0$ is an exact sequence. Hence, $N = M$, which is absurd. So $F \otimes N \neq F \otimes M$. Now, by Theorem 2.4, $F \otimes N$ is a prime submodule of $F \otimes M$.

Conversely, suppose that $F \otimes N$ is a prime submodule of $F \otimes M$. We have, $F \otimes N \neq F \otimes M$ and, obviously, $N \neq M$. Since F is a flat R-module, $F \otimes \frac{M}{N} \cong \frac{F\otimes M}{F\otimes N}$. By Lemma 2.3, the natural homomorphism $h_r : \frac{F\otimes M}{F\otimes N} \to \frac{F\otimes M}{F\otimes N}$, defined by $h_r((f \otimes m) + F \otimes N) = r(f \otimes m) + F \otimes N$, is one-to-one or zero. If $h_r : \frac{F\otimes M}{F\otimes N} \to \frac{F\otimes M}{F\otimes N}$ is zero then the homomorphism $g_r : F \otimes \frac{M}{N} \to F \otimes \frac{M}{N}$, defined by $g_r(f \otimes (m+N)) = f \otimes (rm+N)$, is zero. If we consider $f_r : \frac{M}{N} \to \frac{M}{N}$, given by $f_r(m+N) = rm+N$, then $0 = g_r(F \otimes \frac{M}{N}) = F \otimes f_r(\frac{M}{N})$. Now, since F is faithfully flat, then $f_r(\frac{M}{N}) = 0$, that is $f_r = 0$. If $h_r : \frac{F\otimes M}{F\otimes N} \to \frac{F\otimes M}{F\otimes N}$ is a monomorphism, then $g_r : F \otimes \frac{M}{N} \to F \otimes \frac{M}{N}$ is a monomorphism and again, since F is faithfully flat, $f_r : \frac{M}{N} \longrightarrow \frac{M}{N}$ is a monomorphism. Therefore, N is a prime submodule of M, by Lemma 2.3.

Corollary 2.6 i) *Let F be a flat R-module and I a prime ideal of R. If $IF \neq F$, then IF is a prime submodule of F.*

ii) *If F is a faithfully flat R-module, and I is an ideal of R, then I is a prime ideal of R if and only if IF is a prime submodule of F.*

Proof Having that $IF \cong F \otimes I$, we put $M = R$. Now, the proof is clear by Theorem 2.4 and Proposition 2.5.

Proposition 2.7 *For a ring R the following statements are equivalent*:

i) *The PIT holds for every finitely generated R-module.*

ii) *For every prime ideal P of R, the PIT holds for any finitely generated $\frac{R}{P}$-module.*

iii) *For every prime ideal P of R, the GPIT holds for any finitely generated $\frac{R}{P}$-module.*

iv) *The GPIT holds for every finitely generated R-module.*

Proof i) $\Longrightarrow$ ii) Let B be a cyclic submodule of a finitely generated $\frac{R}{P}$-module M and let N be a minimal prime submodule of M over B. It is obvious that N is a minimal prime submodule over the cyclic submodule B of M as an R-module. So, ht $N \leq 1$ as an R-submodule of M and, obviously, ht $N \leq 1$ as an $\frac{R}{P}$-submodule of M.

ii) $\Longrightarrow$ iii) Since $\frac{R}{P}$ is an integral domain, the proof is completed by [5, Proposition 18].

iii) $\Longrightarrow$ iv) Let M be a finitely generated R-module, B be a submodule of M, which is generated by n elements, and let N be a minimal prime submodule over B. If ht $N > n$, then there exists a chain of prime submodules of M,

$$N_{n+1} \subset N_n \subset N_{n-1} \subset \cdots \subset N_1 \subset N_0 = N.$$

Let $(N_{n+1} : M) = P$. One can easily check that $\frac{N}{N_{n+1}}$ is a minimal prime submodule over the submodule $\frac{B+N_{n+1}}{N_{n+1}}$, which is generated by n elements in the finitely generated $\frac{R}{P}$-module $\frac{M}{N_{n+1}}$. So, by iii), ht $\frac{N}{N_{n+1}} \leq n$. Now by considering the chain $\frac{N_{n+1}}{N_{n+1}} \subset \frac{N_n}{N_{n+1}} \subset \frac{N_{n-1}}{N_{n+1}} \subset \cdots \subset \frac{N_1}{N_{n+1}} \subset \frac{N}{N_{n+1}}$ of prime submodules, we get a contradiction.

iv) $\Longrightarrow$ i) The proof is clear.

Lemma 2.8 *If R is a Noetherian domain, then the PIT holds for every finitely generated R-module if and only if R is a Dedekind domain.*

Proof See [5, Theorem 13].

Proposition 2.9 i) *Let R be a Noetherian ring. Then the GPIT holds for every finitely generated R-module if and only if, for every prime ideal P of R, $\frac{R}{P}$ is a Dedekind domain.*

ii) *If R is a ZPI-ring, or an almost multiplication ring, then the GPIT holds for every finitely generated R-module.*

Proof i) The proof follows from Proposition 2.7 and Lemma 2.8.

ii) If R is a ZPI-ring, $\frac{R}{P}$ is a Dedekind domain for every prime ideal P of R (see [6, p. 205]). Now the proof follows from part i). If R is an almost multiplication ring, then by Theorem 9.23 of [6], for every prime ideal P of R, R_P is a ZPI-ring. The result follows from Lemma 2.1 and Corollary 2.2.

Recall that, if R is an integral domain with the quotient field K, then the rank of an R-module M, which is written as rank M or $\text{rank}_R\, M$, is the maximal number of elements of M linearly independent over R (see, [7, p. 84]). Indeed rank M is equal to the dimension (rank) of the vector space KM over the field K; i.e., rank $M = \text{rank}_K\, KM$ (see, [8, Lemma 2.12]).

The following theorem is a generalization of Proposition 1, of [9]:

Theorem 2.10 *Let R be an integral domain, M an R-module, B a submodule of M and let N be a prime submodule of M minimal over B. If $(N : M) = 0$, then* ht $N =$ rank B.

Proof Let K be the quotient field of R. Since N is a prime submodule of M minimal over B, and $(N : M) = 0$, then by Lemma 2.1, KN is a minimal prime submodule (subspace) of the K-module (vector space) KM over the submodule (subspace) KB. It is easy to see that in a vector space every proper subspace is prime, then since KB itself is prime, we have $KB = KN$. So by Corollary 2.2, $\text{ht}_R\, N = \text{ht}_K\, KN = \text{ht}_K\, KB$. Again since in a vector space every proper subspace is prime, then obviously $\text{ht}_K\, KB = \text{rank}_K\, KB$. Also we know that $\text{rank}_K\, KB = \text{rank}_R\, B$, thus ht $N =$ rank B.

3 Prime Submodules in Free Modules

Lemma 3.1 *Let M be an R-module, N a submodule of M, and $(N : M) = P$ be a prime ideal of R. Then N is a prime submodule of M if and only if $\frac{M}{N}$ is a torsion-free $\frac{R}{P}$-module.*

Proof The proof is clear (see [1, Theorem 1]).

Proposition 3.2 *If R is a ring, P is a prime ideal of R, and $F = \oplus_{i\in I} R$, then a proper submodule $N = \oplus_{i\in I} J_i$, is a P-prime submodule of F if and only if, for each $i \in I$, $J_i = P$ or $J_i = R$.*

Proof Suppose that $\cap_{i\in I} J_i = L$, where L is a prime ideal of R. It is easy to see that $(N : F) = L$. Since $\frac{F}{N} = \frac{\oplus_{i\in I}R}{\oplus_{i\in I}J_i} \cong \oplus_{i\in I}\frac{R}{J_i}$, then $\frac{F}{N}$ is a torsion-free $\frac{R}{L}$-module if and only if, for each $i \in I$, $\frac{R}{J_i}$ is a torsion-free $\frac{R}{L}$-module. This is equivalent to saying that $J_i = L$ or $J_i = R$, for each $i \in I$. Otherwise, if $J_i \neq L$ and $J_i \neq R$, for some $i \in I$, then since $L \subset J_i$, let $r \in J_i - L$, and $t \in R - J_i$. Now we have $(r + L)(t + J_i) = rt + J_i = 0$, $r + L \neq 0$, and $t + J_i \neq 0$.

Now let N be a P-prime submodule. By Lemma 3.1, $\frac{F}{N}$ is a torsion-free $\frac{R}{P}$-module. So, by the above paragraph for each $i \in I$, $J_i = P$ or $J_i = R$.

Conversely, suppose that, for each $i \in I$, $J_i = P$ or $J_i = R$. Since N is a proper submodule of F, then for at least one $i_0 \in I$, we have $J_{i_0} \neq R$. This means that $J_{i_0} = P$, and therefore $(N : F) = \cap_{i\in I} J_i = P$, which is a prime ideal of R. Since, for each $i \in I$, $J_i = P$ or $J_i = R$, then obviously, for each $i \in I$, $\frac{R}{J_i}$ is a torsion-free $\frac{R}{P}$-module and by the first paragraph $\frac{F}{N}$ is a torsion-free $\frac{R}{P}$-module. Thus by Lemma 3.1, N is a prime submodule of F.

Note A similar proof to that of Proposition 3.2, shows that:

If R is a ring, P is a prime ideal of R and $F = \prod_{i\in I} R$, then a proper submodule $N = \prod_{i\in I} J_i$ is a P-prime submodule of F if and only if for each $i \in I$, $J_i = P$ or $J_i = R$.

If N_1 and N_2 are prime submodules of a module M such that $N_1 \subset N_2$ and there does not exist any prime submodule of M strictly between N_1 and N_2, then we say that the chain $N_1 \subset N_2$ is saturated .

Theorem 3.3 *Let F be a free R-module of finite rank with a basis $\{x_1, x_2, \ldots, x_n\}$ and P be a prime ideal of R. Then*:

i) *For each integer k, $0 \leq k \leq n-1$, $N(k, P) = \sum_{i=1}^{k} Rx_i + \sum_{i=k+1}^{n} Px_i$, is a P-prime submodule of F, and $\mathrm{ht}_P\ N(k, P) = k$,*

ii) *For each integer k, $1 \leq k \leq n-1$, the chain $N(k-1, P) \subset N(k, P)$ is saturated,*

iii) *If $P_1 \subset P_2$ is a chain of prime ideals of R, then for each integer k, $0 \leq k \leq n-1$, there is no P_2-prime submodule of F strictly between $N(k, P_1)$ and $N(k, P_2)$,*

iv) *If $P_1 \subset P_2$ is a saturated chain of prime ideals of R, such that P_2 is a principal ideal of R and $\cap_{m=1}^{+\infty} P_2^m = 0$, then for each integer k, $0 \leq k \leq n-1$, the chain $N(k, P_1) \subset N(k, P_2)$ is saturated,*

v) *For each integer k, $0 \leq k \leq n-1$, $\mathrm{ht}\ N(k, P) \geq k + \mathrm{ht}\ P$.*

Proof i) Put $(\oplus_{i=1}^{k} R) \oplus (\oplus_{i=k+1}^{n} P) = T_k$. Obviously, $F \cong R^{(n)} = \oplus_{i=1}^{n} R$, and $N(k, P) \cong T_k$. So, by Proposition 3.2, $N(k, P)$ is a P-prime submodule of F. Evidently, $\mathrm{ht}_P\ N(k, P) = \mathrm{ht}_P\ T_k$. It is easy to see that $N_1 \subset N_2 \subset \cdots \subset N_m \subset T_k$ is a chain of P-prime submodules of $R^{(n)}$ if and only if $\frac{N_1}{PR^{(n)}} \subset \frac{N_2}{PR^{(n)}} \subset \cdots \subset \frac{N_m}{PR^{(n)}} \subset \frac{T_k}{PR^{(n)}}$ is a chain of prime submodules of $\frac{R^{(n)}}{PR^{(n)}}$ as an $\frac{R}{P}$-module. So, $\mathrm{ht}_P\ T_k = \mathrm{ht}\ \frac{T_k}{PR^{(n)}}$. On the other hand, $(\frac{T_k}{PR^{(n)}} : \frac{R^{(n)}}{PR^{(n)}}) = \frac{(T_k : R^{(n)})}{P} = 0$. Therefore, by Theorem 2.10, we have $\mathrm{ht}\ \frac{T_k}{PR^{(n)}} = \mathrm{rank}_{\frac{R}{P}}\ \frac{T_k}{PR^{(n)}}$. One can easily see that $\frac{T_k}{PR^{(n)}} \cong \oplus_{i=1}^{k} \frac{R}{P}$ as an $\frac{R}{P}$-module, hence $\mathrm{rank}_{\frac{R}{P}}\ \frac{T_k}{PR^{(n)}} = k$.

ii) Let Q be a prime submodule of F strictly between $N(k-1, P)$ and $N(k, P)$. By part i), $N(k-1, P)$ and $N(k, P)$ are P-prime submodules of F, and since $N(k-1, P) \subset Q \subset N(k, P)$, then $P = (N(k-1, P) : F) \subseteq (Q : F) \subseteq (N(k, P) : F) = P$. This shows that Q is a P-prime submodule of F. So, the chain $PF = N(0, P) \subset N(1, P) \subset \cdots \subset N(k-1, P) \subset Q \subset N(k, P)$ is a chain of P-prime submodules of F. This is a contradiction, since, by part i), $\mathrm{ht}_P\ N(k, P) = k$.

iii) Let Q be a P_2-prime submodule of F between $N(k, P_1)$ and $N(k, P_2)$. Without loss of generality we can assume that $F = R^{(n)}$, $N(k, P_1) = (\oplus_{i=1}^{k} R) \oplus (\oplus_{i=k+1}^{n} P_1)$, and $N(k, P_2) = (\oplus_{i=1}^{k} R) \oplus (\oplus_{i=k+1}^{n} P_2)$. Let $x = (x_1, x_2, \ldots, x_k, p_{k+1}, p_{k+2}, \ldots, p_n) \in N(k, P_2)$. Then for each i, $k+1 \leq i \leq n$, $p_i \in P_2$. We now show that $x \in Q$.

Obviously, $y = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) \in N(k, P_1) \subseteq Q$. Since $(Q : F) = P_2$, then $P_2F \subseteq Q$, and hence $z = (0, 0, \ldots, 0, p_{k+1}, p_{k+2}, \ldots, p_n) = p_{k+1}(0, 0, \ldots, 0, 1, 0, \ldots, 0) + p_{k+2}(0, 0, \ldots, 0, 1, 0, \ldots, 0) + \cdots + p_n(0, 0, \cdots, 0, 1) \in Q$. Therefore, $x = y + z \in Q$.

iv) Let Q be a prime submodule of F between $N(k,P_1)$ and $N(k,P_2)$. Since $N(k,P_1) \subseteq Q \subseteq N(k,P_2)$, then $P_1 = (N(k,P_1):F) \subseteq (Q:F) \subseteq (N(k,P_2):F) = P_2$. Hence $(Q:F) = P_1$ or $(Q:F) = P_2$.

If $(Q:F) = P_2$, then by part iii), we have $Q = N(k,P_2)$. Now let $(Q:F) = P_1$. Without loss of generality we can assume that $F = R^{(n)}$, $N(k,P_1) = (\oplus_{i=1}^{k} R) \oplus (\oplus_{i=k+1}^{n} P_1)$, and $N(k,P_2) = (\oplus_{i=1}^{k} R) \oplus (\oplus_{i=k+1}^{n} P_2)$. Suppose that $x = (x_1, x_2, \ldots, x_n) \in Q$. We show that $x \in N(k,P_1)$. Indeed, we will show that, for each $i, i \geq k+1$, $x_i = 0$.

Obviously, we have $y = (x_1, x_2, \ldots, x_k, 0, 0, \ldots, 0) \in N(k,P_1) \subseteq Q$ and $z = x - y = (0, 0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots, x_n) \in Q \subseteq N(k,P_2)$. So, for each i, $k+1 \leq i \leq n$, $x_i \in P_2$. Assume that, for every i satisfying in $k+1 \leq i \leq s$, we have $x_i \neq 0$, and, for every i satisfying $s+1 \leq i \leq n$, we have $x_i = 0$. Since $\cap_{m=1}^{+\infty} P_2^m = 0$, then, for each i, $k+1 \leq i \leq s$, there exists a natural number n_i such that $x_i \in P_2^{n_i} - P_2^{n_i+1}$. Without loss of generality, we can assume that $n_{k+1} = \min\{n_i, k+1 \leq i \leq s\}$. Suppose $P_2 = Rp$. Let, for every i, $k+1 \leq i \leq s$, $x_i = p^{n_{k+1}} t_i$. So, we have $p^{n_{k+1}}(0, 0, \ldots, 0, t_{k+1}, t_{k+2}, \ldots, t_s, 0, 0, \ldots, 0) = z \in Q$, and since Q is a P_1-prime submodule, then $(0, 0, \ldots, 0, t_{k+1}, t_{k+2}, \ldots, t_s, 0, 0, \ldots, 0) \in Q \subseteq N(k,P_2)$. Consequently, $t_{k+1} \in P_2$, which is a contradiction.

Note that in this case $P_1 = 0$. To prove this result, let $w \in P_1$. Thus, $w = r_1 p$, for some $r_1 \in R$. Since $p \notin P_1$, $r_1 \in P_1$, and so $r_1 = r_2 p$ for some $r_2 \in R$. Therefore, $w = r_2 p^2 \in P_2^2$. By induction we can show that $w \in P_2^m$ for each $m \geq 1$. Hence, $w \in \cap_{m=1}^{+\infty} P_2^m = 0$.

v) Let ht $P = m$, and $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_m = P$ be a chain of prime ideals of R. By Proposition 3.2, for each s, $0 \leq s \leq m$, $P_s F$ ($P_s F \cong \oplus_{i=1}^{n} P_s$) is a prime submodule of F, and by part i), the following is a chain of prime submodules of F :

$$P_0 F \subset P_1 F \subset P_2 F \subset \cdots \subset P_m F = PF \subset N(1,P) \subset N(2,P) \subset \cdots \subset N(k,P).$$

Hence ht $N(k,P) \geq k+m$.

Proposition 3.4 *Let R be a principal ideal domain, F a free R-module of finite rank and N a P-prime submodule of F. Then*:

i) *There exist a basis $\{x_1, x_2, \ldots, x_n\}$ of F and an integer k, $0 \leq k \leq n-1$, such that $N = N(k,P)$.*

ii) *For each integer k, $0 \leq k \leq n-1$, the chain $N(k,0) \subseteq N(k,P)$ is saturated.*

iii) *If $P \neq 0$, then* ht $N(k,P) = k+1$.

iv) *If $P = 0$, then* ht $N(k,P) = k$.

Proof i) If $N = 0$, then $N = N(0,0)$. Now let $N \neq 0$. Since R is a principal ideal domain, there exist a basis $\{x_1, x_2, \ldots, x_n\}$ of F and also an integer d, $1 \leq d \leq n$, such that $\{r_1 x_1, r_2 x_2, \ldots, r_d x_d\}$ is a basis of N, where $r_i \in R - \{0\}$, and $r_1 | r_2 | r_3 | \cdots | r_d$. Let $(N:F) = P = Rp$, where p is a prime element of R or $p = 0$.

First we let p be a prime element of R. Then $px_i \in N$ for all $i = 1, 2, \ldots, n$ and therefore, $px_i = r_i' r_i x_i$ for some $r_i' \in R$. Hence $r_i | p$ for all i = 1, 2, $\ldots, n$. Consequently, $r_i = 1$ or $r_i = p$ (up to multiplication by a unit of R). Furthermore, since $px_n \in N$, then in this case $d = n$. Let, for each i, $0 \leq i \leq k$, $r_i = 1$ (up to multiplication by a unit of R), and for each i, $k+1 \leq i \leq n$, $r_i = p$. Then we have $N = N(k,P)$.

Now let $p = 0$. Since $r_i x_i \in N$ and N is a prime submodule of F, then $r_i = 0$, or $x_i \in N$. Note that $x_i \in N$ implies $r_i = 1$. So, in this case, for each i, $r_i = 0$, or $r_i = 1$. Let, for each i, $0 \leq i \leq k$, $r_i = 1$, and for each i, $k+1 \leq i \leq d$, $r_i = 0$. Then we have $N = N(k,0)$.

ii) If $P = 0$, obviously $N(k,0) = N(k,P)$. So let $P \neq 0$. Since R is a principal ideal domain, then P is a principal ideal and $\cap_{m=1}^{+\infty} P_2^m = 0$, and hence we have the result by Theorem 3.3 iv).

iii) Let $B = \sum_{i=1}^{k} Rx_i + Px_{k+1}$ Then obviously, $N(k,0) \subseteq B \subseteq N(k,P) = N$. By part ii), there is no prime submodule of F between $N(k,0)$ and $N(k,P)$. So N is a minimal prime submodule over B, and since R is a principal ideal domain, by Proposition 2.9 i), the GPIT holds for F over R, hence ht $N \leq k+1$. Also, by Theorem 3.3 v), we have ht $N \geq k+1$.

iv) Evidently, rank $N = k$, and since N is a minimal prime submodule over itself, then by Theorem 2.10, we have ht $N =$ rank $N = k$.

Proposition 3.5 *Let R be a principal ideal domain, F a free R-module of finite rank and N a minimal prime submodule over a non-zero cyclic submodule B of F. Then*:

i) ht $N = 1$.

ii) *$N = PF$ for some prime ideal P of R or N is a cyclic submodule.*

Proof By Proposition 2.9 i), the GPIT holds for F over R. Hence ht $N \leq 1$, and since 0 is a prime submodule, then, ht $N = 1$. Therefore, if $(N : F) = 0$, then the number k introduced in Proposition 3.4 iv), is 1. That is, $N = N(1, 0) = Rx_1$. Thus, N is a cyclic submodule.

If $(N : F) \neq 0$, let $(N : F) = P$. So, $0 \subset PF \subseteq N$ and we know that ht $N = 1$. Hence $N = PF$.

References

[1] Lu, C. P.: Prime submodules of modules. *Comment. Math. Univ. St. Paul*, **33**(1), 61–69 (1984)

[2] McCasland, R. L., Moore, M. E.: Prime submodules. *Comm. Algebra*, **20**(6), 1803–1817 (1992)

[3] Lu, C. P.: Spectra of modules. *Comm. Algebra*, **23**(10), 3741–3752 (1995)

[4] Marcelo, A., Masque, M.: Prime submodule, the descent invariant, and module of finite length. *Journal of Algebra*, **189**, 273–293 (1997)

[5] George, A. M., McCasland, R. L., Smith, P. F.: A principal ideal theorem analogue for modules over commutative rings. *Comm. Algebra*, **22**, 2083–2099 (1994)

[6] Larsen, M. D., McCarthy, P. J.: Multiplicative theory of ideals, Academic press, Inc., New York, 1971

[7] Matsumura, H.: Commutative ring theory, Cambridge University Press, Cambridge, 1992

[8] Azizi, A.: Intersectin of prime submodules and dimension of modules. *Acta Math. Scientia*, **25B**(3), 385–394 (2005)

[9] Azizi, A., Sharif, H.: On prime submodules. *Honam Mathematical Journal*, **21**(1), 1–12 (1999)

[10] Azizi, A.: Weak multiplication modules. *Czech Mathematical Journal*, **53**(128), 529–534 (2003)

[11] Sharp, R. Y.: Steps in Commutative Algebra, Cambridge University Press, Cambridge, 1990