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Static Theory for Planar Ferromagnets and Antiferromagnets

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Abstract Here we generalize the "BBH"-asymptotic analysis to a simplified mathematical model for the planar ferromagnets and antiferromagnets. To develop such a static theory is a necessary step for a rigorous mathematical justification of dynamical laws for the magnetic vortices formally derived in [1] and [2].

Keywords Ginzburg-Landau-type equations, Vortices, Minimizing harmonic maps, Gradient estimate, Radial solutions, Stability, Quantization
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1 Introduction

Topological solitons (vortices) arise in a variety of physical problems and have been the subject of much study over the last four decades or so. Among the best known examples are domain walls and magnetic bubbles in a ferromagnetic continuum, vortices in superfluids and superconductors, topological defects in liquid crystals, as well as skyrmions, monopoles and instantons which are particle-like solutions in generic models of high-energy physics. The present work addresses the static theory for some simplified model of planar ferromagnets and antiferromagnets. The motivation of such a study comes from attempting a rigorous mathematical justification of the dynamical laws of magnetic vortices formally derived in [1], [2] etc.

The magnetic vortices have been studied extensively for ferromagnets and weak ferromagnets. In both cases a nonvanishing magnetization develops in the ground state, albeit by a different physical mechanism, which then allows detailed experimental investigations. In contrast, direct experimental evidence for pure antiferromagnetic vortices is absent. Nonetheless,

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theoretical arguments suggest that such vortices should exist for essentially the same reason as in ordinary ferromagnets, even though the governing dynamical equations are sufficiently different.

Though the best known examples of topological magnetic solitons are magnetic bubbles observed in abundance in ferromagnetic films with an easy-axis anisotropy [3], the experimental situation is also relatively less clear in ferromagnets with an easy-plane anisotropy (planar ferromagnets) for which the relevant solitons are theoretically predicted to be half bubbles or vortices. It turns out that the energy functionals controlling the statics of planar ferromagnets and antiferromagnets are essentially the same, see for examples [1], [2] and [4].

To describe this simplified model, we let $\Omega \subset \mathbb{R}^2$ be a bounded open connected domain with smooth boundary. Define $S^1 = \{x \backslash x \in \mathbb{R}^3, x^3 = 0, (x^1)^2 + (x^2)^2 = 1\}$, $S^2 = \{x \backslash x \in \mathbb{R}^3, (x^1)^2 + (x^2)^2 + (x^3)^2 = 1\}$. Let $g : \partial\Omega \to S^1$ be a smooth map of degree d. Let $H_g^1\left(\Omega, S^2\right) = \{u \backslash u \in H^1(\Omega, S^2), u|_{\partial\Omega} = g\}$. For any $u \in H_g^1\left(\Omega, S^2\right)$, we let

$$I_\varepsilon(u) = \int_\Omega \frac{1}{2}\left[|\nabla u|^2 + \frac{(u^3)^2}{\varepsilon^2}\right] dx, \quad \varepsilon > 0. \tag{1.1}$$

The energy functional (1.1) supposedly controls the statics of planar ferromagnets and antiferromagnets. As in [1], [2], [4], we are interested in the behavior of critical points of I_ε as $\varepsilon \to 0^+$. If we replace S^2 by $\mathbb{R}^2$ and $\frac{(u^3)^2}{\varepsilon^2}$ by $\frac{(1-|u|^2)^2}{2\varepsilon^2}$, then the problem becomes the familiar simplified model of the Ginzburg-Landau theory for spercanductors.

For the Ginzburg-Landau energy functional, the asymptotic analysis for minimizers (or even more general critical points) has been carried out in [5] and [6]. There are numerous developments since these works, see the lectures [7] by the second author (on a brief description of the state of the art before 1995), and higher-dimensional analogues. Though our analysis closely follows the seminal work of [6], there are many new subtle difficulties. A simple reason for this is that we are now working with S^2-valued maps. This nonlinear, nonconvex constraint in the variational problem (1.1) gives rise to similar difficulties for the study of harmonic maps. In other words, we have to deal with both infinite energy concentrations and finite energy bubbling. Our first result is the following (see Theorem 2.1):

Theorem 1.1 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded connected open domain with smooth boundary. Let $g : \partial\Omega \to S^1$ be a smooth map of degree 0, and denote*

$$\mathcal{M}_g = \left\{ u \backslash u \in H_g^1\left(\Omega, S^1\right), \int_\Omega |\nabla u|^2 = \inf_{v \in H_g^1(\Omega, S^1)} \int_\Omega |\nabla v|^2 \right\}. \tag{1.2}$$

Then there exists an $\varepsilon_* = \varepsilon_*(g, \Omega) > 0$ such that for any $0 < \varepsilon \le \varepsilon_*(g, \Omega)$, any u_ε minimizes I_ε on $H_g^1\left(\Omega, S^2\right)$, we have $u_\varepsilon^3 = 0$ and $u_\varepsilon \in \mathcal{M}_g$. Moreover, $\mathcal{M}_g$ is a finite set of smooth maps.

We note that Theorem 1.1 is somewhat different from Theorem 1 in [5]. In [5], the minimizers u_ε can only approximate the limiting harmonic map u_0 in the space $C^{1,\alpha}$ for any $\alpha \in (0, 1)$ (not in C^2!). It is also clear that the image of u_ε in their case can not be in S^1 except for that it is a constant. The reason for this difference is because S^1 is a totally geodesic submanifold in S^2 but not in $\mathbb{R}^2$.

When the degree $d \neq 0$, as in [6], the minimum energy is going to ∞ as $\varepsilon \to 0^+$. We have (see Theorem 4.1):

Theorem 1.2 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded open simply connected domain with smooth boundary. Let $g : \partial\Omega \to S^1$ be a smooth map with $\deg(g, \partial\Omega, S^1) = d > 0$. For a sequence u_{ε_i}, minimizers of I_{ε_i} on $H^1_g(\Omega, S^2)$, $\varepsilon_i \to 0^+$, after taking a subsequence if necessary, there exist d distinct points $a_1, \ldots, a_d \in \Omega$ such that*

$$u_{\varepsilon_i} \to u_* = \left(\prod_{j=1}^{d} \frac{x - a_j}{|x - a_j|} e^{ih_a(x)}, 0 \right) \quad \text{in } C^\infty_{\text{loc}}(\overline{\Omega} \backslash \{a_1, \ldots, a_d\}).$$

Here h_a is harmonic in Ω and $u_*|_{\partial\Omega} = g$. Moreover, for $\delta > 0$ small, $x \in \Omega \backslash \cup_{i=1}^{d} B_\delta(a_i)$ and $k \in \mathbb{Z}$, $k \geq 0$, we have

$$|D^k u^3_\varepsilon(x)| \leq c(k, \delta, g, \Omega) \, e^{-\frac{1}{c(k,\delta,g,\Omega)\varepsilon_i}}.$$

More information about the locations of the so-called vortex points $a_1, \ldots, a_d$ and some precise asymptotic formulas will be described in Theorem 4.2.

One of the key points involved in applying "BBH"-asymptotic analysis to our problem is the following gradient estimate (see Theorem 3.1):

Theorem 1.3 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded open domain with smooth boundary, and suppose $g : \partial\Omega \to S^1$ is smooth. Then there exist $\varepsilon_* = \varepsilon_*(g, \Omega) > 0$, $c = c(g, \Omega) > 0$ such that for any $0 < \varepsilon \leq \varepsilon_*(g, \Omega)$, any u_ε minimizing I_ε, we have $|\nabla u_\varepsilon(x)| \leq \frac{c(g,\Omega)}{\varepsilon}$ for $x \in \overline{\Omega}$.*

Indeed this gradient estimate is also true for solutions with an image lying in a half-sphere (see Proposition 6.3), which need not be a minimizer. Theorem 1.3 is the starting point in the proof of Theorem 1.2 (even though one may use other arguments, see [8]). We may use the techniques in [6] after having this gradient estimate. Moreover, it gives a better understanding of minimizers.

The study of the existence and stability of special solutions to the Euler-Lagrange equation (2.1) of I_ε would also be quite helpful for understanding the dynamics of vortices. Especially for the problem

$$-\triangle u = \left(|\nabla u|^2 + \frac{(u^3)^2}{\varepsilon^2} \right) u - \frac{u^3}{\varepsilon^2} e_3, \quad \text{on } B^2_1, \quad u(x) = \left(e^{iq\theta}, 0 \right) \quad \text{for } x \in \partial B_1, \qquad (1.3)$$

here $u \in C^\infty\left(\overline{B^2_1}, S^2\right)$, $q \in \mathbb{N}$. We have the following (see Proposition 5.1, Proposition 5.3 and Proposition 5.4):

Proposition 1.1 *There exists a unique $f = f_{\varepsilon,q}$ defined on $[0,1]$ such that $f(0) = 0$, $f(1) = \frac{\pi}{2}$ and $u_{\varepsilon,q} = \left(\sin f(r) \, e^{iq\theta}, \cos f(r) \right)$ is a smooth solution to (1.3). In addition, f satisfies $0 < f(t) \leq \frac{\pi}{2}$, $f'(t) > 0$ for $0 < t \leq 1$. For any $\varepsilon > 0$, $u_{\varepsilon,1}$ is strictly stable, and hence a local minimizer. If $q \geq 2$, then for $0 < \varepsilon < \varepsilon(q)$, $u_{\varepsilon,q}$ is unstable.*

The existence of solutions in Proposition 1.1 is dealt with by reducing (1.3) to ordinary differential equations. The stability result is studied along the same lines as the work of [9] for the Ginzburg-Landau model. However, there are again some new technical difficulties. Due to the nonlinearity introduced by S^2, the second variation formula contains certain first-order terms, unlike the one in [9]. These difficulties are overcome by a careful study of the qualitative property of $f_{\varepsilon,q}$ in the above proposition.

One of the most interesting aspects of our problem is the so-called energy quantization phenomenon similarly to [10]. If we have a map $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ satisfying

$$-\triangle u = \left(|\nabla u|^2 + \left(u^3\right)^2\right) u - u^3 e_3 \tag{1.4}$$

on $\mathbb{R}^2$, then do we have $\frac{1}{\pi}\int_{\mathbb{R}^2}\left(u^3\right)^2 \in \mathbb{Z}$? If u decays fast enough at ∞, then this is the case. In fact we have (see Proposition 6.1):

Proposition 1.2 *Suppose $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ satisfies (1.4) on $\mathbb{R}^2$, $u^3 \to 0$ as $|x| \to \infty$ and there exists $c > 0$ such that*

$$\int_{B_r}\left(|\nabla u|^2 + \left(u^3\right)^2\right) \le c\log r \quad \textit{for} \quad r \ge 2. \tag{1.5}$$

Then $\int_{\mathbb{R}^2}\left(u^3\right)^2 = \pi d^2$, where d is the degree of $\frac{u'}{|u'|}$ at ∞, $u' = \left(u^1, u^2\right)$. Moreover

$$|D^k u^3(x)| \le c(k,u)\, e^{-c|x|},\ c > 0,\ c(k,u) > 0, \quad \textit{for any } k \ge 0,$$

$$|D^k u(x)| \le \frac{c(k,u)}{|x|^k} \textit{ for } k \ge 1.$$

If we write $u^1 + iu^2 = \rho e^{i(d\theta+\psi)}$ outside ball B_{R_0}, then $|\nabla\psi(x)| = O\left(\frac{1}{|x|^2}\right)$.

We point out that if $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ locally minimizes I_1, then it satisfies the growth conditions in Proposition 1.2 (see [11]) and hence the quantization property is correct. Another important case for such quantization to be valid is when the image of u lies in the closed upper half sphere. That is (see Proposition 6.4):

Theorem 1.4 *Suppose $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ satisfies (1.4) on $\mathbb{R}^2$, $u^3 \ge 0$, $\liminf_{|x|\to\infty}|\nabla u(x)| = 0$, $\int_{\mathbb{R}^2}\left(u^3\right)^2 dx < \infty$. Then either $u^3 \equiv 0$ or*

$$|u^3(x)| \le c(u)e^{-\frac{|x|}{16}},\ |\nabla u^3(x)| \le c(u)e^{-\frac{|x|}{16}},\ |\nabla u(x)| \le \frac{c(u)}{|x|},\ \int_{\mathbb{R}^2}\left(u^3\right)^2 dx = \pi d^2,$$

where d is the degree of $\frac{u'}{|u'|}$ at ∞, $u' = \left(u^1, u^2\right)$.

We shall present various examples so that such energy quantization may be false. It is obvious that there are many more rich classes of entire solutions of (1.4) than the one studied in [10].

In our forthcoming works (also jointly with Jalal Shatah), we should apply the static theory developed here to the study of the dynamics of magnetic vortices.

The paper is written as follows: In Section 2 below, we prove that when the degree of g is zero and ε is small enough, then the minimizer of I_ε is in fact a minimizing harmonic map to S^1. Section 3 proves the gradient estimate for minimizers. In Section 4, we use the gradient estimate proved in Section 3 and arguments from [6] to prove the convergence of minimizers away from $|d|$ vortex points. We also present some asymptotic formulas for this convergence. Section 5 studies special solutions to the Euler-Lagrange equation of I_ε and the stability property of these solutions. In Section 6 we present several results on the energy quantization.

After the present work was accepted, we learned from S. Serfaty an earlier work by N. Andre and I. Shafrir: "On nematics stabilized by a large external field", Rev. in Math Phys., Vol. 11, No. 6, 1999, 653–710. In that paper authors studied many similar issues. However, we noticed that one of the key point in the proof, the gradient estimates (see our Theorem 1.3), was not fully explained and verified. They also did not discuss these energy quatization results as well as Liouville type theorems.

2 The Case $\deg(g, \partial\Omega, S^1) = 0$

In this section, we shall study the behavior of minimizers of I_ε as $\varepsilon \to 0^+$ for the case $\deg\left(g, \partial\Omega, S^1\right) = 0$. Before we proceed, we would like to establish some basic properties for minimizers.

Lemma 2.1 *There exists at least one $u_\varepsilon \in H^1_g(\Omega, S^2)$ which minimizes I_ε. All minimizers are smooth and satisfy*

$$-\triangle u_\varepsilon = \left(|\nabla u_\varepsilon|^2 + \frac{(u^3_\varepsilon)^2}{\varepsilon^2}\right) u_\varepsilon - \frac{u^3_\varepsilon}{\varepsilon^2} e_3 \quad \text{in} \quad \Omega, \quad u_\varepsilon|_{\partial\Omega} = g. \tag{2.1}$$

Here $e_3 = (0, 0, 1)$.

Proof The existence follows from the direct method in the calculus of variations. The smoothness of minimizers follows from [12]. In fact, it follows from [13] and [14] that every critical point of I_ε is smooth.

Lemma 2.2 *Suppose u_ε is a minimizer of I_ε. Then either $u^3_\varepsilon \equiv 0$ in Ω or $u^3_\varepsilon > 0$ in Ω or $u^3_\varepsilon < 0$ in Ω.*

Proof Let u_ε be a minimizer of I_ε. Putting $v(x) = (u^1_\varepsilon(x), u^2_\varepsilon(x), |u^3_\varepsilon(x)|)$, then $v \in H^1_g(\Omega, S^2)$ is also a minimizer. From Lemma 2.1 we know v is smooth and it satisfies

$$-\triangle v^3 = \left(|\nabla v|^2 + \frac{(v^3)^2}{\varepsilon^2}\right) v^3 - \frac{v^3}{\varepsilon^2}, \quad v^3 \geq 0, \ v^3|_{\partial\Omega} = 0.$$

It follows from the Harnack inequality that either $v^3 \equiv 0$ or $v^3 > 0$ in Ω. If $v^3 \equiv 0$, then $u^3_\varepsilon \equiv 0$. If $v^3 > 0$ in Ω, then $u^3_\varepsilon > 0$ in Ω or $u^3_\varepsilon < 0$ in Ω.

Now we may state the main theorem of this section:

Theorem 2.1 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded connected open domain with smooth boundary. Let $g : \partial\Omega \to S^1$ be a smooth map of degree 0, and denote*

$$\mathcal{M}_g = \left\{ u \backslash u \in H^1_g(\Omega, S^1), \int_\Omega |\nabla u|^2 = \inf_{v \in H^1_g(\Omega, S^1)} \int_\Omega |\nabla v|^2 \right\}. \tag{2.2}$$

Then there exists an $\varepsilon_* = \varepsilon_*(g,\Omega) > 0$ such that for any $0 < \varepsilon \le \varepsilon_*(g,\Omega)$, any u_ε minimizes I_ε on $H^1_g(\Omega, S^2)$, we have $u^3_\varepsilon = 0$ and $u_\varepsilon \in \mathcal{M}_g$. Moreover $\mathcal{M}_g$ is a finite set of smooth maps.

Remark 2.1 We note that in Theorem 1 of [5], the minimizers u_ε can only approximate the harmonic map u_0 in the space $C^{1,\alpha}$ for any $\alpha \in (0,1)$. It is also clear that the image of u_ε in their case can't be in S^1 except when it is a constant. This difference of our result from theirs is due to a simple geometric fact that S^1 is a totally geodesic submanifold of S^2 but not of $\mathbb{R}^2$.

We use the idea in Lecture 1 of [7] to show Theorem 2.1.

Lemma 2.3 *Suppose g, Ω are as in Theorem 2.1. Then $\mathcal{M}_g$ is non-empty and compact in $H^1(\Omega)$.*

Proof Since the degree of g is zero, we may find a smooth extension $\tilde{u} : \overline{\Omega} \to S^1$, then from the direct method in the calculus of variations we know $\mathcal{M}_g$ is non-empty. Put

$$\lambda = \inf_{v \in H^1_g(\Omega, S^1)} \int_\Omega |\nabla v|^2. \tag{2.3}$$

Suppose $u_j \in \mathcal{M}_g$. Then $\int_\Omega |\nabla u_j|^2 = \lambda$, which implies that we may find a subsequence which is still denoted as u_j such that $u_j \rightharpoonup u$ for some $u \in H^1(\Omega)$. Hence $u \in H^1_g(\Omega, S^1)$ and

$$\int_\Omega |\nabla u|^2 \le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 = \lambda. \tag{2.4}$$

From (2.4) we get $\int_\Omega |\nabla u|^2 = \lambda$, $u_j \to u$ in $H^1(\Omega)$ and $u \in \mathcal{M}_g$.

Corollary 2.1 *Under the assumptions of Theorem 2.1, we have for any $\varepsilon > 0$, there exists $\delta = \delta(g,\Omega,\varepsilon) > 0$ such that for $u \in \mathcal{M}_g$ and $E \subset \Omega$, we have $\int_E |\nabla u|^2 \le \varepsilon$ whenever $|E| \le \delta$.*

Lemma 2.4 *Under the assumptions of Theorem 2.1, for any $\varepsilon_0 > 0$, there exists an $r_0 = r_0(g,\Omega,\varepsilon_0) > 0$ such that for every minimizer u_ε and every $x \in \overline{\Omega}$ we have*

$$\int_{B_{r_0}(x)\cap\Omega} |\nabla u_\varepsilon|^2 + \frac{(u^3_\varepsilon)^2}{\varepsilon^2} \le \varepsilon_0,$$

if $0 < \varepsilon \le \varepsilon_*(g,\Omega,\varepsilon_0)$.

Proof From Corollary 2.1 we know there exists an $r_0 = r_0(g,\Omega,\varepsilon_0) > 0$ such that for $x \in \overline{\Omega}$ and $u \in \mathcal{M}_g$,

$$\int_{B_{r_0}(x)\cap\Omega} |\nabla u|^2 \le \frac{\varepsilon_0}{2}. \tag{2.5}$$

If the conclusion of Lemma 2.4 is false, then there would exist $\varepsilon_j \to 0$, $u_j = u_{\varepsilon_j}$ minimizing I_{ε_j} and $x_j \in \overline{\Omega}$ such that

$$\int_{B_{r_0}(x_j)\cap\Omega} |\nabla u_j|^2 + \frac{(u^3_j)^2}{\varepsilon^2_j} > \varepsilon_0. \tag{2.6}$$

After passing to a subsequence we may assume $x_j \to x_*$. On the other hand, for every $v \in H^1_g(\Omega, S^1)$, we have

$$\int_\Omega |\nabla u_j|^2 + \frac{(u^3_j)^2}{\varepsilon^2_j} \le \int_\Omega |\nabla v|^2. \tag{2.7}$$

After passing to a subsequence again we may assume $u_j \rightharpoonup w$ in $H^1(\Omega)$, $u_j \to w$ in $L^2(\Omega)$. $\int_\Omega (u_j^3)^2 \le \varepsilon_j^2 \int_\Omega |\nabla v|^2 \to 0$ implies $w^3 = 0$, hence $w \in H_g^1(\Omega, S^1)$. Also from $\int_\Omega |\nabla w|^2 \le \int_\Omega |\nabla v|^2$, we know $\int_\Omega |\nabla w|^2 = \lambda$ and $w \in \mathcal{M}_g$. $\mathcal{M}_g$ and λ are defined in (2.2) and (2.3). Taking $v = w$ in (2.7), we have

$$\int_\Omega |\nabla u_j|^2 + \frac{(u_j^3)^2}{\varepsilon_j^2} \le \int_\Omega |\nabla w|^2, \tag{2.8}$$

$$\int_\Omega |\nabla w|^2 \le \liminf_{j\to\infty} \int_\Omega |\nabla u_j|^2 \le \limsup_{j\to\infty} |\nabla u_j|^2 \le \int_\Omega |\nabla w|^2. \tag{2.9}$$

Hence $\int_\Omega |\nabla u_j|^2 \to \int_\Omega |\nabla w|^2$, the latter fact implies $u_j \to w$ in $H^1(\Omega)$. Going back to (2.8) we get $\int_\Omega \frac{(u_j^3)^2}{\varepsilon_j^2} \to 0$. Gathering all these facts, we have

$$\begin{aligned}\int_{B_{r_0}(x_j)\cap\Omega} |\nabla u_j|^2 + \frac{(u_j^3)^2}{\varepsilon_j^2} &= \int_{B_{r_0}(x_j)\cap\Omega} |\nabla w|^2 + \int_{B_{r_0}(x_j)\cap\Omega} (|\nabla u_j|^2 - |\nabla w|^2) \\ &\quad + \int_{B_{r_0}(x_j)\cap\Omega} \frac{(u_j^3)^2}{\varepsilon_j^2} \to \int_{B_{r_0}(x_*)\cap\Omega} |\nabla w|^2. \end{aligned}\tag{2.10}$$

(2.6) and (2.10) together imply $\int_{B_{r_0}(x_*)\cap\Omega} |\nabla w|^2 \ge \varepsilon_0$; this contradicts (2.5) because $w \in \mathcal{M}_g$. Hence the conclusion of Lemma 2.4 follows.

Lemma 2.5 *Suppose $u : \Omega \to S^2$ is smooth and $-\triangle u = (|\nabla u|^2 + \frac{(u^3)^2}{\varepsilon^2})u - \frac{u^3}{\varepsilon^2} e_3$, where $e_3 = (0,0,1)$. Denoting $e_\varepsilon(u) = \frac{1}{2}\left[|\nabla u|^2 + \frac{(u^3)^2}{\varepsilon^2}\right]$, then*

$$-\triangle e_\varepsilon(u) = 4e_\varepsilon(u)^2 - |D^2 u|^2 - \frac{2}{\varepsilon^2}|\nabla u^3|^2 - \frac{(u^3)^2}{\varepsilon^4}, \tag{2.11}$$

and hence $-\triangle e_\varepsilon(u) \le 4e_\varepsilon(u)^2$.

Proof For each $k = 1, 2$ or 3, we have

$$-\triangle u^k = 2e_\varepsilon(u)u^k - \frac{u^3}{\varepsilon^2}\delta_3^k, \quad \triangle e_\varepsilon(u) = |D^2 u|^2 + \sum_{i,j} \partial_i u^j \partial_i \triangle u^j + \frac{1}{\varepsilon^2}(|\nabla u^3|^2 + u^3 \triangle u^3);$$

plugging in the equation of u, we get the conclusion.

Lemma 2.6 *If $v \in C^\infty(\overline{B_r})$, $v \ge 0$, $-\triangle v \le v^2$ on B_r, then there exists an $\eta_0 > 0$, such that $\int_{B_r} v \le \eta_0$ implies $\sup_{B_{\frac{r}{2}}} v \le c \fint_{B_r} v \le \frac{c}{r^2}\eta_0$. Here c and η_0 are absolute constants.*

Proof By scaling we may assume $r = 1$. Put $K = \max_{|x|\le 1}(1-|x|)^2 v(x)$. We claim $K \le 1$. Otherwise, if $K > 1$, choosing $x_0 \in B_1$ such that $(1-|x_0|)^2 v(x_0) = K$. Setting $\sigma = 1 - |x_0|$, then for $x \in B_{\frac{\sigma}{2}}(x_0)$, $v(x) \le \frac{4K}{\sigma^2}$. $w(x) = \frac{\sigma^2}{4K} v(x_0 + \frac{\sigma}{2\sqrt{K}}x)$ is well defined on B_1. It satisfies

$$-\triangle w \le w^2, \quad w \le 1 \text{ on } B_1, \quad w(0) = \frac{1}{4}, \quad \int_{B_1} w = \int_{B_{\frac{\sigma}{2\sqrt{K}}}(x_0)} v(x)dx \le \eta_0.$$

Hence $-\triangle w \le w$. From the mean value inequality we know $w(0) \le c \fint_{B_1} w \le c\eta_0$. Here c is an absolute constant. Choose η_0 small enough such that $c\eta_0 < \frac{1}{4}$, and we get a contradiction.

Hence $K \le 1$. On $B_{\frac{3}{4}}$ we have $v(x) \le 16$, $-\triangle v \le 16v$. So again by the mean value inequality we get $v(x) \le c \int_{B_{\frac{3}{4}}} v \le c \int_{B_1} v \le c\eta_0$.

Corollary 2.2 *Under the assumptions of Theorem* 2.1*, there exists an* $\varepsilon_* = \varepsilon_*(g, \Omega) > 0$ *such that for* $0 < \varepsilon \le \varepsilon_*(g, \Omega)$*, any* $u_\varepsilon \in H^1_g(\Omega, S^2)$ *minimizing* I_ε*, and* K *any compact subset of* Ω*, we have*

$$\sup_K \left(|\nabla u_\varepsilon|^2 + \frac{(u^3_\varepsilon)^2}{\varepsilon^2} \right) \le c(g, \Omega, K).$$

Proof This follows from Lemma 2.4, Lemma 2.5 and Lemma 2.6.

Now we estimate up to the boundary.

Lemma 2.7 *Under the assumption of Theorem* 2.1*, there exists an* $\varepsilon_* = \varepsilon_*(g, \Omega) > 0$ *such that for any* $0 < \varepsilon \le \varepsilon_*(g, \Omega)$ *and any* u_ε *minimizing* I_ε*, we have*

$$\sup_\Omega \left(|\nabla u_\varepsilon|^2 + \frac{(u^3_\varepsilon)^2}{\varepsilon^2} \right) \le c(g, \Omega).$$

Proof From Lemma 2.2 we may assume $u^3_\varepsilon \ge 0$. We prove Lemma 2.7 by a contradiction argument. If Lemma 2.7 is false, then there would exist $\varepsilon_j \to 0$, $u_j = u_{\varepsilon_j}$ minimizing I_{ε_j} such that

$$\sup_\Omega \left(|\nabla u_j|^2 + \frac{(u^3_j)^2}{\varepsilon^2_j} \right) = K_j \to \infty. \tag{2.12}$$

Suppose the maximum is reached at $x_j \in \overline{\Omega}$. From Corollary 2.2 we know x_j must go to $\partial\Omega$, we may assume $x_j \to x_*$ for some $x_* \in \partial\Omega$. By the arguments in Lemma 2.4, after passing to a subsequence, there exists a $u \in \mathcal{M}_g$ such that $u_j \to u$ in $H^1(\Omega)$ and $\int_\Omega \frac{(u^3_j)^2}{\varepsilon^2_j} \to 0$. Denote x^*_j as the closest point to x_j on $\partial\Omega$. By rotation and translation, we may assume $x^*_j = 0$ and the tangent line of $\partial\Omega$ is the coordinate line $\{x^2 = 0\}$, hence $x^1_j = 0$. Put

$$\tau_j = \sqrt{K_j}\varepsilon_j > 0, \quad v_j(x) = u_j\left(\frac{x}{\sqrt{K_j}}\right), \quad y_j = \sqrt{K_j} x_j.$$

For $e_{\tau_j}(v_j) = \frac{1}{2}\left[|\nabla v_j|^2 + \frac{(v^3_j)^2}{\tau^2_j}\right]$, we have $e_{\tau_j}(v_j)(y_j) = \frac{1}{2}$ and $e_{\tau_j}(v_j) \le \frac{1}{2}$. First we observe $y^2_j \to 0$, otherwise we may use Lemma 2.6 to get a contradiction. Then we observe that τ_j must go to zero, otherwise we may use standard elliptic estimates and the fact that $\int_{B_1 \cap \Omega_j} e_{\tau_j}(v_j) \to 0$ (Ω_j is the domain of v_j) to get a contradiction. From $v^3_j \le \tau_j$ we know $\sup_{\Omega_j} v^3_j \to 0$. Putting $v^1_j + iv^2_j = \rho_j e^{i\psi_j}$ on $B_2 \cap \Omega_j$ then ρ_j is very close to 1 and we have

$$\operatorname{div}(\rho^2_j \nabla\phi_j) = 0, \quad \triangle\rho_j - \rho_j|\nabla\phi_j|^2 = -2\rho_j e_{\tau_j}(v_j). \tag{2.13}$$

Since ϕ_j is smooth on $\partial\Omega_j \cap B_2$ and converges to a constant, we have

$$|\nabla\phi_j|_{C^\alpha(B_1\cap\Omega_j)} \to 0. \tag{2.14}$$

Setting $r_j = 1 - \rho_j$, then we have

$$r_j|_{\partial\Omega_j \cap B_2} = 0,\ r_j \ge 0,\ -\triangle r_j \le \rho_j|\nabla\phi_j|^2 \le \delta_j \to 0 \text{ in } B_1 \cap \Omega_j. \tag{2.15}$$

We also know $r_j \to 0$ uniformly. A barrier argument tells us

$$|\nabla \rho_j|_{L^\infty(\partial\Omega_j \cap B_{\frac{1}{2}})} \le \alpha_j, \tag{2.16}$$

where $\alpha_j \to 0$. Since

$$-\triangle v_j^3 \le v_j^3, \quad v_j^3|_{\partial\Omega_j \cap B_2} = 0, \quad v_j^3 \to 0 \text{ uniformly}, \tag{2.17}$$

similar arguments tell us

$$|\nabla v_j^3|_{L^\infty(\partial\Omega_j \cap B_{\frac{1}{2}})} \le \beta_j, \tag{2.18}$$

where $\beta_j \to 0$. From (2.14), (2.16) and (2.18) we get $e_{\tau_j}(v_j) \le \gamma_j$, $\gamma_j \to 0$ on $\partial\Omega_j \cap B_{\frac{1}{2}}$. Now by the mean value inequality near the boundary (see Chapter 8 of [15]) and $-\triangle e_{\tau_j}(v_j) \le 2e_{\tau_j}(v_j)$ in Ω_j, we get

$$\frac{1}{2} = e_{\tau_j}(v_j)(y_j) \le c\left(\int_{B_{\frac{1}{2}} \cap \Omega_j} e_{\tau_j}(v_j) + \gamma_j\right) \to 0,$$

which is a contradiction.

Proof of Theorem 2.1 By Lemma 2.7 we know for $0 < \varepsilon \le \varepsilon_*(g,\Omega)$ and any u_ε minimizing I_ε, we have

$$\sup_\Omega \left(|\nabla u_\varepsilon|^2 + \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2}\right) \le c_0(g,\Omega). \tag{2.19}$$

We may assume $\varepsilon_*(g,\Omega) \le \frac{1}{\sqrt{c_0(g,\Omega)}}$; then for $0 < \varepsilon \le \varepsilon_*(g,\Omega)$, we have

$$\frac{1}{\varepsilon^2} - \left(|\nabla u_\varepsilon|^2 + \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2}\right) \ge 0. \tag{2.20}$$

Now since

$$-\triangle u_\varepsilon^3 + \left(\frac{1}{\varepsilon^2} - \left(|\nabla u_\varepsilon|^2 + \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2}\right)\right) u_\varepsilon^3 = 0, \quad u_\varepsilon^3|_{\partial\Omega} = 0, \tag{2.21}$$

it follows from the maximum principle and (2.20) that $u_\varepsilon^3 \equiv 0$, then it is clear that $u_\varepsilon \in \mathcal{M}_g$. To see $\mathcal{M}_g$ is finite, we define

$$\mathcal{H}_g = \left\{u \backslash u \in H_g^1\left(\Omega, S^1\right) \text{ is a harmonic map to } S^1\right\}. \tag{2.22}$$

Fix a $u_0 \in \mathcal{M}_g$, then the map from $\mathcal{H}_1$ (defined in the same way as in (2.22) by replacing g with 1) to $\mathcal{H}_g$ given by $u \mapsto (u_0 \cdot u)$ (complex multiplication) is a bijection. Let $\partial\Omega = \cup_{j=0}^n C_j$, $C_0, \ldots, C_n$ be connected components of $\partial\Omega$. Then for any $u \in \mathcal{H}_1$, $u = e^{i\varphi}$ with $\varphi \in C^\infty\left(\overline{\Omega}, \mathbb{R}\right)$, a harmonic function satisfying $\varphi|_{C_0} = 0$, $\varphi|_{C_j} = 2k_j\pi$, $k_j \in \mathbb{Z}$ for $1 \le j \le n$. Let $\varphi_j \in C^\infty\left(\overline{\Omega}, \mathbb{R}\right)$ be the harmonic function on Ω with $\varphi_j|_{C_k} = \delta_{jk}$, for $0 \le k \le n$. Then

$$\varphi = \sum_{j=1}^n 2k_j\pi\varphi_j. \tag{2.23}$$

Since $\nabla\varphi_1,\ldots,\nabla\varphi_n$ are linearly independent, we may find a real number $c>0$, such that for any $a_1,\ldots,a_n\in\mathbb{R}$,

$$\int_\Omega |\nabla\left(a_1\varphi_1+\cdots+a_n\varphi_n\right)|^2\geq c\sum_{j=1}^n a_j^2. \tag{2.24}$$

For the $u\in\mathcal{H}_1$ above, by (2.5) and (2.6) we have

$$\int_\Omega|\nabla u|^2=\int_\Omega|\nabla\varphi|^2\geq c\sum_{j=1}^n k_j^2. \tag{2.25}$$

For any $v\in\mathcal{M}_g$, $v=u_0\cdot u$ with $u\in\mathcal{H}_1$, then for the φ, k_j corresponding to u, we have

$$c\sum_{j=1}^n k_j^2\leq\int_\Omega|\nabla u|^2\leq 2\int_\Omega|\nabla u_0|^2+2\int_\Omega|\nabla v|^2\leq 2\int_\Omega|\nabla u_0|^2+2\lambda, \tag{2.26}$$

where λ is defined in (2.3). (2.26) implies $\#\mathcal{M}_g<\infty$.

3 A Gradient Estimate for Minimizers

In this section we shall prove a gradient estimate for minimizers.

Theorem 3.1 *Suppose $\Omega\subset\mathbb{R}^2$ is a bounded open domain with smooth boundary, and suppose $g:\partial\Omega\to S^1$ is smooth. Then there exist $\varepsilon_*=\varepsilon_*(g,\Omega)>0$, $c=c(g,\Omega)>0$ such that for any $0<\varepsilon\leq\varepsilon_*(g,\Omega)$, any u_ε minimizing I_ε, we have $|\nabla u_\varepsilon(x)|\leq\frac{c(g,\Omega)}{\varepsilon}$ for $x\in\overline{\Omega}$.*

We shall prove in Proposition 6.3 that this gradient estimate is true for all solutions lying in a half sphere, which need not be a minimizer. Both proofs will be helpful for the future development. We also note that the map g in Theorem 3.1 is not necessarily of degree zero.

We need to establish several Liouville-type theorems before proving the gradient estimate.

Lemma 3.1 *Suppose u is a continuous subharmonic function on $\mathbb{R}^2$ which is bounded from above. Then it is a constant.*

Proof This is a well-known fact for $\mathbb{R}^2$ which is not true for $\mathbb{R}^n$, $n>2$. The reason for this difference is because the fundamental solution of the Laplacian in two dimensions is essentially different from higher-dimensional ones. One may prove the lemma by a simple comparison with the logarithm function.

Lemma 3.2 *Suppose $u:\mathbb{R}^2\to S^2$ is a smooth harmonic map with $u^3\geq 0$. Then either $u\equiv\mathrm{const}$ or $u(x)=(\cos\psi(x),\sin\psi(x),0)$, where ψ is a harmonic function on $\mathbb{R}^2$.*

Proof From the harmonic map equation we know $-\triangle u^3=|\nabla u|^2u^3\geq 0$. Hence $-u^3$ is subharmonic on $\mathbb{R}^2$. Because it is bounded, from Lemma 3.1 we conclude that $u^3\equiv c$, a constant. If $c>0$, then $c|\nabla u|^2=-\triangle c=0$ implies u is a constant. If $c=0$, then $u^3=0$ and the image of u is in S^1. Since $\mathbb{R}^2$ is simply connected, we know $u(x)=(\cos\psi(x),\sin\psi(x),0)$ and ψ is a harmonic function.

The following is an easy calculation:

Lemma 3.3 *Suppose U is an open subset in an m dimensional Riemannian manifold M, $u \in C^\infty\left(U, S^{n-1}\right)$, $\varphi \in C_c^\infty(U, \mathbb{R}^n)$, define $u(x,t) = \frac{u(x)+t\varphi(x)}{|u(x)+t\varphi(x)|}$, $\phi(t) = \int_U |\nabla_M u(x,t)|^2 d\mu_M$. Then*

$$\phi''(0) = 2\int_U \left[\langle \nabla\left(3(u\cdot\varphi)^2 u - 2(u\cdot\varphi)\varphi - |\varphi|^2 u\right), \nabla u\rangle + |\nabla\left(\varphi - (u\cdot\varphi)u\right)|^2\right] d\mu_M.$$

If, in addition, we know u is a harmonic map, then

$$\phi''(0) = 2\int_U \left[|\nabla\left(\varphi^T\right)|^2 - |\nabla u|^2|\varphi^T|^2\right] d\mu_M.$$

Here $\varphi^T = \varphi - (\varphi\cdot u)\,u$. If, in addition, $\phi''(0) \geq 0$ for all $\varphi \in C_c^\infty(U, \mathbb{R}^n)$, then we say u is stable.

Lemma 3.4 *If $u : \mathbb{R}^2 \to S^2$ is a smooth stable harmonic map with $u^3 \geq 0$, then it is a constant map. In particular, any locally minimizing harmonic map from $\mathbb{R}^2$ to a half sphere is a constant map.*

Proof If u is not a constant, then from Lemma 3.2 we know $u(x) = (\cos\psi(x), \sin\psi(x), 0)$ and ψ is a nonconstant real harmonic function. From Lemma 3.3 we know that for any $\eta \in C_c^\infty\left(\mathbb{R}^2, \mathbb{R}\right)$, by taking $\varphi = (0,0,\eta)$, we have

$$\int_{\mathbb{R}^2} |\nabla\eta|^2 - \eta^2|\nabla u|^2 \geq 0. \tag{3.1}$$

Fix an $\eta_0 \in C_c^\infty\left(\mathbb{R}^2, \mathbb{R}\right)$ such that $0 \leq \eta_0 \leq 1$, $\eta_0|_{B_1} = 1$, $\eta_0|_{\mathbb{R}^2\setminus B_2} = 0$. In (3.1), setting $\eta(x) = \eta_R(x) = \eta_0\left(\frac{x}{R}\right)$ for $R > 0$ and letting $R \to \infty$, we get $\int_{\mathbb{R}^2} |\nabla\psi|^2 \leq c$, an absolute constant. Hence $\psi \equiv$ const, a contradiction.

We shall prove in [11] that any minimizing harmonic map from $\mathbb{R}^2$ to S^2 is a constant map; here S^2 need not have the standard metric.

Remark 3.1 The condition $u^3 \geq 0$ in Lemma 3.4 can't be dropped because any holomorphic or anti-holomorphic map from $\mathbb{R}^2$ to S^2 is stable. In fact, a theorem of A. Lichnerowicz says every holomorphic or anti-holomorphic map from a compact Kähler manifold to another Kähler manifold is energy minimizing in its homotopy class (see Theorem 4.2 in [16]). If we examine the proof closely, one can easily show that without the compactness condition on the domain manifold, any holomorphic or anti-holomorphic map is energy minimizing in its homotopy class if we only consider those homotopies supported in a compact subset. In particular, this shows holomorphic or anti-holomorphic maps between Kähler manifolds are always stable harmonic maps.

We will use Lemma 3.4 to classify all the blowing-up maps of certain equations later. Indeed we only need the following version, which is slightly different from the above one. We present it here because the proof will be quite helpful for further development.

Lemma 3.5 *Let $u(x) = (e^{i\left(c_0 + c_1 x^1 + c_2 x^2\right)}, 0)$, where c_0, c_1, c_2 are real constants, either c_1 or c_2 being nonzero. Then u is not locally minimizing for I_1 (see (1.1) for definition) on $\mathbb{R}^2$.*

Proof By contradiction. If u is locally minimizing I_1, without loss of generality we may assume $u(x) = (e^{i\lambda x^1}, 0)$, $\lambda > 0$. Choose a map $w : \left[0, \frac{2\pi}{\lambda}\right] \times [0,1] \to S^2$ such that w is Lipschitz and

$$w(0,t) = w\left(\frac{2\pi}{\lambda}, t\right) = w(s,1) = (1,0,0), \quad w(s,0) = \left(e^{i\lambda x^1}, 0\right), \text{ for} 0 \le t \le 1,\ 0 \le s \le \frac{2\pi}{\lambda}.$$

Consider for $l > 0$, $I_l = \left\{x \backslash 0 \le x^1 \le \frac{2\pi}{\lambda},\ 0 \le x^2 \le l+2\right\}$. For $x \in I_l$, define

$$v_l(x) = \begin{cases} w(x) & \text{if } \ 0 \le x^2 \le 1, \\ (1,0,0) & \text{if } \ 1 \le x^2 \le l+1, \\ w\left(x^1, l+2-x^2\right) & \text{if } \ l+1 \le x^2 \le l+2. \end{cases}$$

Then $v_l|_{\partial I_l} = u|_{\partial I_l}$, hence $I_1(u) \le I_1(v_l)$. In other words, $2\pi\lambda(l+2) \le 2I_1(w)$. Letting $l \to \infty$, we get a contradiction. Another way to prove the lemma is the following: If u is locally minimizing, then for every $R > 1$, define $v_R : B_R \to \mathbb{C}$ by

$$v_R(x) = \begin{cases} (R - |x|) + (|x| - R + 1)\, e^{i\left(c_0 + c_1 x^1 + c_2 x^2\right)} & \text{if } \ R-1 \le |x| \le R, \\ 1 & \text{if } \ |x| \le R-1. \end{cases}$$

It follows from formula (5.11) that $I_1(\Gamma \circ v_R, B_R) \le c(c_1, c_2) R$. Here Γ is the stereographic projection defined in (4.1). Hence $\frac{1}{2}\left(c_1^2 + c_2^2\right) R^2 = I_1(u, B_R) \le c(c_1, c_2) R$. Letting $R \to \infty$, we get a contradiction.

Lemma 3.6 *Denote $H_0 = \{x \backslash x \in \mathbb{R}^2, x^2 > 0\}$, the open upper half plane. Suppose $u : \overline{H_0} \to S^2$ is a smooth harmonic map, which is stable in H_0. If, in addition, $u^3 \ge 0$, $u|_{\partial H_0} = \text{const}$, $|\nabla u(x)| \le 1$, then $u \equiv \text{const}$ in $\overline{H_0}$.*

Proof We have $-\triangle u(x) = |\nabla u(x)|^2 u(x)$. For any sequence $h_j \to \infty$ and l_j, define $u_j(x) = u(x^1 + l_j, x^2 + h_j)$, for $x \in \overline{H_j}$, $H_j = \{x \backslash x \in \mathbb{R}^2, x^2 > -h_j\}$. Then $-\triangle u_j(x) = |\nabla u_j(x)|^2 u_j(x)$, $|\nabla\ u_j(x)| \le 1$ in H_j. Hence for any $\alpha \in (0,1)$ and $r > 0$, $|u_j|_{C^{1,\alpha}(\overline{B_r})} \le c(\alpha, r)$. From the Schauder theory we know after passing to a subsequence $u_j \to v$ in $C^\infty(\mathbb{R}^2)$, $v \in C^\infty\left(\mathbb{R}^2, S^2\right)$. Since u_j is stable, from Lemma 3.3 we know for any $\varphi \in C_c^\infty\left(H_j, \mathbb{R}^3\right)$,

$$\int_{H_j} \left(|\nabla\left(\varphi - (u_j \cdot \varphi)\varphi\right)|^2 - \left(|\varphi|^2 - (u_j \cdot \varphi)^2\right)|\nabla u_j|^2\right) dx \ge 0.$$

Letting $j \to \infty$, we get for any $\varphi \in C_c^\infty\left(\mathbb{R}^2, \mathbb{R}^3\right)$,

$$\int_{H} \left(|\nabla\left(\varphi - (v \cdot \varphi)\varphi\right)|^2 - \left(|\varphi|^2 - (v \cdot \varphi)^2\right)|\nabla v|^2\right) dx \ge 0,$$

that is, v is a stable harmonic map on $\mathbb{R}^2$. Furthermore, $u_j^3 \ge 0$ implies $v^3 \ge 0$. From Lemma 3.4 we know v must be a constant. Hence $\nabla u_j(x) \to 0$ in $C^\infty(\mathbb{R}^2)$. This tells us that

$$\lim_{x^2 \to \infty} \left(\sup_{x^1 \in \mathbb{R}} |\nabla u(x^1, x^2)|\right) = 0. \tag{3.2}$$

Now let us look at the Hopf function

$$\varphi(z) = |\partial_1 u|^2 - |\partial_2 u|^2 - 2i(\partial_1 u \cdot \partial_2 u).$$

We know it is a holomorphic function on the upper half plane (see [17] Section 1). $\mathrm{Im}\varphi$ is harmonic and it is zero on $\{x^2 = 0\}$ because u is constant on this line. It is also bounded because $|\nabla u|$ is. Hence $\mathrm{Im}\varphi = 0$. From the Cauchy-Riemann equations we know $\mathrm{Re}\varphi \equiv c$, a constant. From the limit in (3.2) we know $c = 0$. Hence $|\partial_1 u|^2 = |\partial_2 u|^2$. Since u is a constant on $\{x^2 = 0\}$, we know

$$\partial_1 u(x^1, 0) = \partial_2 u(x^1, 0) = 0. \tag{3.3}$$

Put $\tilde{u}(x) = u(x)$ for $x^2 \geq 0$ and $\tilde{u}(x) = u(x^1, -x^2)$ for $x^2 \leq 0$. Then from (3.3) we know $\tilde{u}$ is a harmonic map on $\mathbb{R}^2$ with $\tilde{u}^3 \geq 0$. If $\tilde{u}$ is not a constant, from Lemma 3.2 we know $\tilde{u}(x) = (\cos\psi(x), \sin\psi(x), 0)$ and ψ is a harmonic function. From the bound of $|\nabla u|$ we know ψ must be linear and $|\nabla \tilde{u}| = |\nabla \psi| \equiv \alpha > 0$. This contradicts the limit in (3.2). Hence $\tilde{u}$ is a constant and we get the lemma.

Proof of Theorem 3.1 Without loss of generality, we assume $u_\varepsilon^3 \geq 0$, by Lemma 2.2. Suppose the conclusion of Theorem 3.1 is not true; then we may find $\varepsilon_j \to 0$, $u_j = u_{\varepsilon_j}$ minimizing I_{ε_j} such that

$$K_j = \varepsilon_j \sup_{x \in \overline{\Omega}} |\nabla u_j(x)| \to \infty.$$

Choose $x_j \in \overline{\Omega}$ such that $\varepsilon_j |\nabla u_j(x_j)| = K_j$, define

$$v_j(x) = u_j\left(x_j + \frac{\varepsilon_j}{K_j}x\right) \text{ for } x \in \overline{\Omega_j},\ \Omega_j = \frac{K_j}{\varepsilon_j}(\Omega - x_j). \tag{3.4}$$

Then

$$-\triangle v_j = \left(|\nabla v_j|^2 + \frac{(v_j^3)^2}{K_j^2}\right) v_j + \frac{v_j^3}{K_j^2} e_3 \text{ on } \Omega_j,\ |\nabla v_j(x)| \leq 1,\ |\nabla v_j(0)| = 1. \tag{3.5}$$

There are two cases we are going to discuss. The first case is $\Omega_j \to \mathbb{R}^2$ as $j \to \infty$. In this case, from (3.5) we get for any $\alpha \in (0,1)$, any $r > 0$, $|v_j|_{C^{1,\alpha}(\overline{B_r})} \leq c(\alpha, r)$. Hence we may assume $v_j \to v$ in $C^\infty(\mathbb{R}^2)$ after passing to a subsequence. We have $v \in C^\infty\left(\mathbb{R}^2, S^2\right)$ and $|\nabla v(0)| = 1$.

Claim 3.1 *v is a locally minimizing harmonic map on the whole plane.*

Proof of Claim 3.1 In fact, for any $r > 0$, $w \in H^1(B_r, S^2)$ such that $w|_{\partial B_r} = v|_{\partial B_r}$. For $0 < \delta < 1$, set

$$w_{j,\delta}(x) = \begin{cases} w\left(\dfrac{x}{1-\delta}\right) & \text{when} \quad |x| \leq (1-\delta)r, \\ \Pi\left(\dfrac{r-|x|}{r\delta} v\left(r\dfrac{x}{|x|}\right) + \dfrac{|x| - r + r\delta}{r\delta} v_j\left(r\dfrac{x}{|x|}\right)\right) & \text{when} \quad (1-\delta)r \leq |x| \leq r, \end{cases}$$

where $\Pi(\xi) = \frac{\xi}{|\xi|}$ for $\xi \in \mathbb{R}^3$. Set

$$w_\delta(x) = \begin{cases} w\left(\dfrac{x}{1-\delta}\right) & \text{when} \quad |x| \leq (1-\delta)r, \\ v\left(r\dfrac{x}{|x|}\right) & \text{when} \quad (1-\delta)r \leq |x| \leq r. \end{cases}$$

We have $w_{j,\delta} \to w_\delta$ in $H^1(B_r)$. Since $w_{j,\delta}|_{\partial B_r} = v_j|_{\partial B_r}$, we know

$$\int_{B_r} |\nabla w_{j,\delta}|^2 + \frac{(w_{j,\delta}^3)^2}{K_j^2} \ge \int_{B_r} |\nabla v_j|^2 + \frac{(v_j^3)^2}{K_j^2}.$$

Letting $j \to \infty$, we get $\int_{B_r} |\nabla w_\delta|^2 \ge \int_{B_r} |\nabla v|^2$. Letting $\delta \to 0$, we have $\int_{B_r} |\nabla w|^2 \ge \int_{B_r} |\nabla v|^2$. This proves Claim 3.1.

Now $v_j^3 \ge 0$ implies $v^3 \ge 0$. It follows from Lemma 3.4 that v is a constant. This contradicts $|\nabla v(0)| = 1$.

The second case is $\Omega_j \to H$, H being a half plane. After rotation we may assume $H = \{x \backslash x \in \mathbb{R}^2, x^2 > -a\}$, where a is a nonnegative number. Since on $\partial\Omega_j$ the v_j is simply a dilation of g, we have for any $r > 0$, $\alpha \in (0,1)$, $|v_j|_{C^{1,\alpha}\left(\overline{B_r \cap \Omega_j}\right)} \le c(\alpha, r, g, \Omega)$. Hence $v_j \to v$ in $C^\infty\left(\overline{H}\right)$, $v \in C^\infty\left(\overline{H}, S^2\right)$. It is constant on ∂H. A similar argument to that above shows v is a locally minimizing harmonic map in H, also $v^3 \ge 0$, $|\nabla v(0)| = 1$. From Lemma 3.6 we know v is a constant, and we have a contradiction.

4 The Case $\deg(g, \partial\Omega, S^1) \ne 0$

In this section, we shall discuss the case when we have a topological obstruction, that is, the case when $\deg\left(g, \partial\Omega, S^1\right) \ne 0$.

Theorem 4.1 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded open simply connected domain with smooth boundary. Let $g : \partial\Omega \to S^1$ be a smooth map with $\deg(g, \partial\Omega, S^1) = d > 0$. For a sequence u_{ε_i}, minimizers of I_{ε_i} on $H_g^1(\Omega, S^2)$, $\varepsilon_i \to 0^+$, after taking a subsequence if necessary, there exist d distinct points $a_1, \ldots, a_d \in \Omega$ such that*

$$u_{\varepsilon_i} \to u_* = \left(\prod_{j=1}^d \frac{x - a_j}{|x - a_j|} e^{ih_a(x)}, 0\right) \quad \text{in } C_{\mathrm{loc}}^\infty(\overline{\Omega} \backslash \{a_1, \ldots, a_d\}).$$

Here h_a is harmonic in Ω and $u_*|_{\partial\Omega} = g$. Moreover, for $\delta > 0$ small, $x \in \Omega \backslash \cup_{i=1}^d B_\delta(a_i)$ and $k \in \mathbb{Z}$, $k \ge 0$, we have

$$|D^k u_\varepsilon^3(x)| \le c(k, \delta, g, \Omega)\, e^{-\frac{1}{c(k,\delta,g,\Omega)\varepsilon_i}}.$$

Remark 4.1 We note that the convergence of minimizers u_ε to u_* away from the vortices is in C^∞ topology, unlike the $C^{1,\alpha}$ convergence in [6]'s case. The reason for this difference is explained in Remark 2.1.

We shall determine the location of $a_1, \ldots, a_d$ after proving this theorem. Recall the following important annulus lemma proved in [10]:

Lemma 4.1 (Annulus Lemma from [10]) *$A = A_{r_0, r_1} = B_{r_1} \backslash \overline{B_{r_0}}$, $u \in H^1(A, \mathbb{R}^2)$, $|u| \ge \sigma > 0$, $\frac{1}{r_0^2}\int_A \left(1 - |u|^2\right)^2 \le K$, $d = \deg\left(\frac{u}{|u|}, \partial B_r\right)$ for $r_0 < r < r_1$, $u = \rho e^{i(d\theta + \psi)}$, where $\rho = |u|$, $\psi \in H^1(A, \mathbb{R})$ is a well-defined function. Then*

$$\int_A |\nabla u|^2 \ge 2\pi d^2 \log\frac{r_1}{r_0} + \int_A |\nabla \rho|^2 + \frac{\sigma^2}{2}\int_A |\nabla\psi|^2 - \left(\sqrt{\pi} + \frac{2d^2}{\sigma^2}\right) K.$$

Lemma 4.2 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded open simply connected domain with smooth boundary, and suppose $\tilde{g}_\varepsilon : \partial\Omega \to S^2$ satisfies*

$$\frac{1}{\varepsilon^2}\int_{\partial\Omega}\left(\tilde{g}_\varepsilon^3\right)^2 ds + |\tilde{g}_\varepsilon|_{H^1(\partial\Omega)} \le K,$$

and $\tilde{g}_\varepsilon \to \tilde{g}$ uniformly on $\partial\Omega$. Here $\deg(\tilde{g}, \partial\Omega, S^1) = 0$, $\tilde{g} = \left(e^{i\varphi_0}, 0\right)$ on $\partial\Omega$. We denote by φ_0 the harmonic extension of itself from $\partial\Omega$ to Ω, $\tilde{u}_0 = \left(e^{i\varphi_0}, 0\right)$ on $\overline{\Omega}$. Suppose $\tilde{u}_\varepsilon$ minimizes I_ε in $H^1_{\tilde{g}_\varepsilon}(\Omega, S^2)$. Then

$$\tilde{u}_\varepsilon \to \tilde{u}_0 \text{ in } H^1(\Omega), \quad \frac{1}{\varepsilon^2}\int_\Omega \left(\tilde{u}_\varepsilon^3\right)^2 \to 0.$$

Proof Denote

$$\Gamma : \mathbb{R}^2 \to S^2\backslash\{(0,0,-1)\}, \quad \Gamma(y^1, y^2) = \left(\frac{2y^1}{1+|y|^2}, \frac{2y^2}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2}\right). \tag{4.1}$$

This is the standard stereographic projection. If we use Γ^{-1} as the coordinate, then the metric on S^2 is given by

$$g_{S^2} = \frac{4}{\left(1+|y|^2\right)^2}\left(dy^1 \otimes dy^1 + dy^2 \otimes dy^2\right).$$

Denote $g_\varepsilon = \Gamma^{-1} \circ \tilde{g}_\varepsilon$, $g = \Gamma^{-1} \circ \tilde{g}$. We want to construct a comparison function $v_\varepsilon = \eta_\varepsilon e^{i\varphi_\varepsilon}$. For convenience, let $d(x) = \operatorname{dist}(x, \partial\Omega)$ for any $x \in \mathbb{R}^2$. There exists a $\delta > 0$ such that for any x with $d(x) < \delta$, there exists a unique $\phi(x) \in \partial\Omega$ such that $|x - \phi(x)| = d(x)$. For $x \in \overline{\Omega}$, let

$$\eta_\varepsilon(x) = \begin{cases} \dfrac{d(x)}{\varepsilon} + \left(1 - \dfrac{d(x)}{\varepsilon}\right)|g_\varepsilon(\phi(x))|, & \text{if } \ d(x) \le \varepsilon; \\ 1, & \text{if } \ d(x) \ge \varepsilon. \end{cases} \tag{4.2}$$

It is clear that $\eta_\varepsilon \to 1$ uniformly on $\overline{\Omega}$. Simple computations show

$$\int_\Omega |\nabla\eta_\varepsilon|^2 \le c(K,\Omega)\varepsilon, \quad \frac{1}{\varepsilon^2}\int_\Omega (1-\eta_\varepsilon)^2 \le c(K,\Omega)\varepsilon. \tag{4.3}$$

On the other hand, on $\partial\Omega$, we may write $g_\varepsilon = |g_\varepsilon|e^{i\varphi_\varepsilon}$ such that $\varphi_\varepsilon \to \varphi_0$ uniformly. We denote by φ_ε the harmonic extension of itself from $\partial\Omega$ to Ω. It follows from the interpolation inequality that $\varphi_\varepsilon \to \varphi_0$ in $H^1(\Omega)$. By considering the minimum property of $\tilde{u}_\varepsilon$ and choosing $\Gamma \circ v_\varepsilon$ as a comparison map we get

$$\begin{aligned} \int_\Omega |\nabla\tilde{u}_\varepsilon|^2 + \frac{1}{\varepsilon^2}\int_\Omega \left(\tilde{u}_\varepsilon^3\right)^2 &\le \int_\Omega \left[\frac{4}{\left(1+|v_\varepsilon|^2\right)^2}|\nabla v_\varepsilon|^2 + \frac{1}{\varepsilon^2}\left(\frac{1-|v_\varepsilon|^2}{1+|v_\varepsilon|^2}\right)^2\right] \\ &= \int_\Omega \left[\frac{4}{\left(1+\eta_\varepsilon^2\right)^2}\left(|\nabla\eta_\varepsilon|^2 + \eta_\varepsilon^2|\nabla\varphi_\varepsilon|^2\right) + \frac{1}{\varepsilon^2}\left(\frac{1-\eta_\varepsilon^2}{1+\eta_\varepsilon^2}\right)^2\right] \\ &\le c(K,\Omega)\varepsilon + \int_\Omega \frac{4\eta_\varepsilon^2}{\left(1+\eta_\varepsilon^2\right)^2}|\nabla\varphi_\varepsilon|^2. \end{aligned} \tag{4.4}$$

Suppose $\tilde{u}_{\varepsilon_i} \rightharpoonup \tilde{u}$ in $H^1(\Omega)$. Taking a limit in (4.4) we get

$$\int_\Omega |\nabla\tilde{u}|^2 \le \int_\Omega |\nabla\varphi_0|^2 = \int_\Omega |\nabla\tilde{u}_0|^2. \tag{4.5}$$

Since $\tilde{u} \in H^1_{\tilde{g}}(\Omega, S^1)$, (4.5) implies $\tilde{u} = \tilde{u}_0$. Note that it follows again from (4.4) that

$$\limsup_{\varepsilon \to 0^+} \int_\Omega |\nabla \tilde{u}_\varepsilon|^2 \le \int_\Omega |\nabla \tilde{u}_0|^2. \tag{4.6}$$

(4.6) implies $\tilde{u}_\varepsilon \to \tilde{u}_0$ in $H^1(\Omega)$. Going back to (4.4), we get $\frac{1}{\varepsilon^2} \int_\Omega \left(\tilde{u}^3_\varepsilon\right)^2 \to 0$.

Lemma 4.3 *Under the assumption of Theorem* 4.1, *there exists a* $c = c(g, \Omega) > 0$ *such that for any* $0 < \varepsilon \le 1$, *any* u_ε *minimizing* I_ε *we have*

$$I_\varepsilon(u_\varepsilon) \le \pi d \log \frac{1}{\varepsilon} + c(g, \Omega). \tag{4.7}$$

Proof Choose a $w \in C^\infty\left(\overline{B_1}, S^2\right)$ such that $w(x) = (x, 0)$ for $x \in \partial B_1$. Pick up d different points in Ω, namely $a_1, \ldots, a_d$, fix a $\rho > 0$ suitably small, let $\Omega_\rho = \Omega \backslash \bigcup_{j=1}^d \overline{B_\rho(a_j)}$. Define $\tilde{g} : \partial\Omega \to S^1$ as

$$\tilde{g}(x) = \begin{cases} g(x) & x \in \partial\Omega, \\ \left(\dfrac{x - a_j}{|x - a_j|}, 0\right) & x \in \partial B_\rho(a_j). \end{cases}$$

Since $\deg(\tilde{g}, \partial\Omega_\rho, S^1) = 0$, we may find $\tilde{u} : \overline{\Omega_\rho} \to S^1$ smooth and $\tilde{u}|_{\partial\Omega_\rho} = \tilde{g}$. Now define

$$v_\varepsilon(x) = \begin{cases} \tilde{u}(x) & x \in \Omega_\rho, \\ \left(\dfrac{x - a_j}{|x - a_j|}, 0\right) & x \in \overline{B_\rho(a_j)} \backslash B_\varepsilon(a_j), \\ w\left(\dfrac{x - a_j}{\varepsilon}\right) & x \in B_\varepsilon(a_j). \end{cases}$$

Then $I_\varepsilon(u_\varepsilon) \le I_\varepsilon(v_\varepsilon) \le \pi d \log \frac{1}{\varepsilon} + c(g, \Omega)$.

Now let us state the Pohozaev identity:

Lemma 4.4 *Suppose* $D \subset \mathbb{R}^2$ *is a bounded open domain with piecewisely* C^1 *boundary. Let* $u \in C^\infty(\overline{D}, S^2)$ *satisfy* $-\triangle u = \left(|\nabla u|^2 + \frac{\left(u^3\right)^2}{\varepsilon^2}\right) u - \frac{u^3}{\varepsilon^2} e_3$. *Then*

$$\begin{aligned} \frac{1}{\varepsilon^2} \int_D \left(u^3\right)^2 + \frac{1}{2} \int_{\partial D} (x \cdot \nu) |\partial_\nu u|^2 ds = {} & \frac{1}{2} \int_{\partial D} (x \cdot \nu) |\partial_\tau u|^2 ds \\ & + \frac{1}{2\varepsilon^2} \int_{\partial D} (x \cdot \nu) \left(u^3\right)^2 ds - \int_{\partial D} (x \cdot \tau) (\partial_\nu u \cdot \partial_\tau u) \, ds, \end{aligned}$$

where ν *is the unit outward normal and* τ *is the unit tangential vector in the positive direction. Suppose* D *is strictly star-shaped with respect to* 0, *i.e. there exist* $\alpha > 0$, $\rho > 0$ *such that* $|x| < \rho$ *for* $x \in D$ *and* $x \cdot \nu \ge \alpha\rho$ *for* $x \in \partial D$. *Then*

$$\frac{1}{\varepsilon^2} \int_D \left(u^3\right)^2 + \frac{\alpha\rho}{4} \int_{\partial D} |\partial_\nu u|^2 ds \le \frac{\rho}{2\varepsilon^2} \int_{\partial D} \left(u^3\right)^2 ds + \left(\frac{\rho}{\alpha} + \frac{\rho}{2}\right) \int_{\partial D} |\partial_\tau u|^2 ds.$$

Proof Multiply the equation by $x^j \partial_j u$, then do integration by parts.

We also need the following comparison function from Lemma 2 on p. 130 of [5]:

Lemma 4.5 *For $R>0$, $\varepsilon>0$, $\omega_\varepsilon(x)=e^{\frac{|x|^2-R^2}{4\varepsilon R}}$, then*

$$-\triangle\omega_\varepsilon+\frac{\omega_\varepsilon}{\varepsilon^2}=\frac{1}{\varepsilon^2}\left(1-\frac{\varepsilon}{R}-\frac{|x|^2}{4R^2}\right)e^{\frac{|x|^2-R^2}{4\varepsilon R}}.$$

In particular, if $0<\varepsilon<\frac{3}{4}R$, then $-\triangle\omega_\varepsilon+\frac{\omega_\varepsilon}{\varepsilon^2}\geq 0$ on B_R.

Proof of Theorem 4.1 First we have

$$\int_\Omega|\nabla u_\varepsilon^3|^2+\frac{1}{\varepsilon^2}\int_\Omega\left(u_\varepsilon^3\right)^2\leq c(g,\Omega)\quad\text{ for any }\quad 0<\varepsilon\leq 1. \tag{4.8}$$

In fact, if we denote $u'_\varepsilon=\left(u_\varepsilon^1,u_\varepsilon^2\right)$, then it follows from [18] and [19] that

$$\begin{aligned}\int_\Omega\frac{1}{2}|\nabla u'_\varepsilon|^2+\frac{\left(u_\varepsilon^3\right)^2}{4\varepsilon^2}dx&=\int_\Omega\frac{1}{2}|\nabla u'_\varepsilon|^2+\frac{1-|u'_\varepsilon|^2}{4\varepsilon^2}dx\\&\geq\int_\Omega\frac{1}{2}|\nabla u'_\varepsilon|^2+\frac{\left(1-|u'_\varepsilon|^2\right)^2}{4\varepsilon^2}dx\geq\pi d\log\frac{1}{\varepsilon}-c(g,\Omega).\end{aligned}$$

This inequality and (4.7) imply (4.8).

To simplify the notations, we will use u_ε to denote u_{ε_i}. By Lemma 2.2, we may assume $u_\varepsilon^3\geq 0$. Denote $S_\varepsilon=\{x\backslash x\in\overline{\Omega},u_\varepsilon^3(x)\geq\frac{1}{2}\}$. For $x\in S_\varepsilon$, we may have $\lambda_0=\lambda_0(g,\Omega)>0$ such that

$$\int_{B_{\lambda_0\varepsilon}(x)\cap\Omega}\frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2}\geq\frac{1}{c(g,\Omega)}>0. \tag{4.9}$$

This follows from the gradient estimate in Theorem 3.1, $|\nabla u_\varepsilon|\leq\frac{c(g,\Omega)}{\varepsilon}$ for ε small. $S_\varepsilon\subset\bigcup_{x\in S_\varepsilon}\overline{B_{\lambda_0\varepsilon}(x)}$; from the Vitali covering lemma we know there exist $x_1^\varepsilon,\dots,x_{N_\varepsilon}^\varepsilon\in S_\varepsilon$ such that $S_\varepsilon\subset\bigcup_{i=1}^{N_\varepsilon}\overline{B_{5\lambda_0\varepsilon}(x_i^\varepsilon)}$ and $\overline{B_{\lambda_0\varepsilon}(x_i^\varepsilon)}$ are mutually non-disjoint. This together with the inequalities (4.8) and (4.9) tells us $N_\varepsilon\leq c(g,\Omega)$. After an induction argument we may assume

$$S_\varepsilon\subset\bigcup_{i=1}^{k_\varepsilon}\overline{B_{\lambda\varepsilon}(x_i^\varepsilon)},\ x_i^\varepsilon\in S_\varepsilon,\ k_\varepsilon\leq c\,(g,\Omega),\ \lambda=\lambda(g,\Omega),\ \ |x_i^\varepsilon-x_j^\varepsilon|\geq 5\lambda\varepsilon\text{ for }i\neq j.$$

We may assume $k_\varepsilon\equiv k$ after passing to a subsequence. We also assume $x_i^\varepsilon\to x_{*i}\in\overline{\Omega}$. Let $a_1,\dots,a_l$ be different points in $x_{*1},\dots,x_{*k}$. Choose a smooth, bounded, connected open set Ω' such that $\overline{\Omega}\subset\Omega'$, also fix a smooth map $\tilde{g}:\overline{\Omega'}\backslash\Omega\to S^1$ such that $\tilde{g}|_{\partial\Omega}=g$. Any map $u:\overline{\Omega}\to S^2$ such that $u|_{\partial\Omega}=g$ can be considered as a map on $\overline{\Omega'}$ by setting $u|_{\overline{\Omega'}\backslash\Omega}=\tilde{g}$. Fix a $\delta>0$ such that $\delta<\mathrm{dist}(\overline{\Omega},\mathbb{R}^2\backslash\Omega')$, $\delta<\frac{1}{2}|a_i-a_j|$ for $i\neq j$. When ε is small we have

$$\bigcup_{i=1}^{k}\overline{B_{\lambda\varepsilon}(x_i^\varepsilon)}\subset\bigcup_{i=1}^{l}B_{\frac{\delta}{4}}(a_i).$$

From the gradient estimate we know

$$|\deg\left(u'_\varepsilon,\partial B_{\lambda\varepsilon}(x_i^\varepsilon)\right)|=\left|\frac{1}{2\pi}\int_{\partial B_{\lambda\varepsilon}(x_i^\varepsilon)}\frac{u'_\varepsilon}{|u'_\varepsilon|^2}\wedge\left(u'_\varepsilon\right)_\tau\right|\leq c\,(g,\Omega).$$

Here $u'_\varepsilon = \left(u^1_\varepsilon, u^2_\varepsilon\right)$. By passing to a subsequence we may assume

$$\deg\left(u'_\varepsilon, \partial B_{\lambda\varepsilon}(x^\varepsilon_i)\right) = d_i,\ i = 1, \ldots, k \text{ and } \deg\left(u'_\varepsilon, \partial B_{\frac{\delta}{2}}(x^\varepsilon_i)\right) = \kappa_i,\ i = 1, \ldots, l.$$

Fix an i, set $\Lambda_i = \{j \backslash 1 \le j \le k, x^\varepsilon_j \to a_i\}$. Then $\sum_{j\in\Lambda_i} d_j = \kappa_i$. Define

$$\Omega^{\varepsilon,\delta}_i = B_\delta(a_i) \backslash \bigcup_{j\in\Lambda_i} \overline{B_{\lambda\varepsilon}(x^\varepsilon_j)}.$$

By Lemma 4.1 and an inductive argument we have, for $i = 1, \ldots, l$, that

$$\int_{\Omega^{\varepsilon,\delta}_i} |\nabla u'_\varepsilon|^2 \ge 2\pi|\kappa_i| \log\frac{\delta}{\varepsilon} - c\,(g, \Omega). \tag{4.10}$$

As in [6], we have $\kappa_i \ge 0$, for $i = 1, \ldots, l$. Indeed, from (4.10) we know

$$\sum_{i=1}^{l} \int_{\Omega^{\varepsilon,\delta}} |\nabla u'_\varepsilon|^2 \ge 2\pi \log\frac{\delta}{\varepsilon} \sum_{i=1}^{l} |\kappa_i| - c\,(g, \Omega), \quad \Omega^{\varepsilon,\delta} = \bigcup_{i=1}^{l} \Omega^{\varepsilon,\delta}_i. \tag{4.11}$$

Comparing this inequality with (4.7) and letting $\varepsilon \to 0^+$, we get

$$\sum_{i=1}^{l} |\kappa_i| \le d = \sum_{i=1}^{l} \kappa_i,$$

which implies $\sum_{i=1}^{l} |\kappa_i| - \kappa_i \le 0$, and hence $\kappa_i \ge 0$.

Combining (4.7) and (4.11) we see

$$\int_{\Omega'} |\nabla u^3_\varepsilon|^2 + \int_{\Omega'_\delta} |\nabla u'_\varepsilon|^2 \le 2\pi d \log\frac{1}{\delta} + c\,(g, \Omega),\ \Omega'_\delta = \Omega' \backslash \bigcup_{i=1}^{d} \overline{B_\delta(a_i)}. \tag{4.12}$$

We may assume $u_\varepsilon \rightharpoonup u_*$ in $H^1_{\text{loc}}\left(\Omega' \backslash \{a_1, \ldots, a_l\}\right)$ and $u_\varepsilon \to u_*$ almost everywhere. From $\int_\Omega \left(u^3_\varepsilon\right)^2 \le c\,(g, \Omega)\varepsilon^2$ we get $u^3_* \equiv 0$. Since $\operatorname{div}\left(u'_\varepsilon \wedge \nabla u'_\varepsilon\right) = u'_\varepsilon \wedge \triangle u'_\varepsilon = 0$ in Ω, taking a limit we get $\operatorname{div}\left(u'_* \wedge \nabla u'_*\right) = 0$ in $\Omega \backslash \{a_1, \ldots, a_d\}$. Hence from [13] we know u_* is a smooth harmonic map into S^1 on $\Omega \backslash \{a_1, \ldots, a_d\}$ and $u_*|_{\partial\Omega} = g$.

Next, we verify each $\kappa_i > 0$. In fact, we already know $\kappa_i \ge 0$, if for some i, $\kappa_i = 0$, then choose a $R_0 > 0$ such that $\overline{B_{R_0}(a_i)}$ is contained in Ω' and it doesn't contain other singularities. After passing to a subsequence we may assume for some $R \in (\frac{R_0}{2}, R_0)$,

$$\int_{\partial B_R(a_i)} |\nabla u_\varepsilon|^2 + \frac{\left(u^3_\varepsilon\right)^2}{\varepsilon^2} \le c,$$

for some c independent of ε, $u_\varepsilon \to u_*$ uniformly on $\partial B_R(a_i)$. From Lemma 4.2 we know

$$\frac{1}{\varepsilon^2} \int_{B_R(a_i)} \left(u^3_\varepsilon\right)^2 \to 0,$$

which contradicts (4.9), because $B_R(a_i)$ contains at least one point of S_ε. This shows each κ_i is positive. In fact each κ_i is exactly equal to 1. To see this, we use Lemma 4.1 to obtain

$$\int_{\Omega'_\delta} |\nabla u_*|^2 \ge 2\pi \sum_{i=1}^{l} \kappa_i^2 \log\frac{1}{\delta} - c\,(g, \Omega). \tag{4.13}$$

On the other hand, by the lower semi-continuity we have

$$\int_{\Omega'_\delta} |\nabla u_*|^2 \leq 2\pi d \log \frac{1}{\delta} + c\,(g, \Omega). \tag{4.14}$$

Combining (4.13) and (4.14) then letting $\delta \to 0^+$ we get $\sum_i \kappa_i^2 \leq \sum_i \kappa_i$, hence $\sum_i (\kappa_i^2 - \kappa_i) \leq 0$, and this forces $\kappa_i = 1$, $l = d$. Via Lemma VI.1 of [6, p. 63] and (4.14) we have $a_i \in \Omega$, for $i = 1, \ldots, d$.

From the above arguments we know $u_* = \left(\prod_{j=1}^d \frac{x-a_j}{|x-a_j|} e^{ih(x)}, 0\right)$, where h is a harmonic function in $\Omega \backslash \{a_1, \ldots, a_d\}$, and $u_*|_{\partial\Omega} = g$. Denote $a = (a_1, \ldots, a_d)$. Fixing a suitably small δ_0, we have from the proof of Lemma 4.1 that

$$\int_{A_{\delta,\delta_0}(a_i)} |\nabla u_*|^2 \geq 2\pi \log \frac{\delta_0}{\delta} + \frac{1}{2} \int_{A_{\delta,\delta_0}(a_i)} |\nabla h|^2 - c(g, \Omega, a),$$

for $0 < \delta < \delta_0$. This, together with (4.14), implies

$$\int_{A_{\delta,\delta_0}(a_i)} |\nabla h|^2 \leq c\,(g, \Omega, a), \text{ for } i = 1, \ldots, d.$$

The latter fact implies h is of finite energy on the whole Ω and hence it is harmonic in Ω and fully determined by its boundary value. We call it h_a, then $u_* = \left(\prod_{j=1}^d \frac{x-a_j}{|x-a_j|} e^{ih_a(x)}, 0\right)$. Now pick up any point $x \in \overline{\Omega} \backslash \{a_1, \ldots, a_d\}$, and for suitably small $R > 0$,

$$\int_{B_R(x)\cap\Omega} \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2} \to 0, \; u_\varepsilon \to u_* \text{ in } H^1\left(B_R(x) \cap \Omega\right).$$

This, together with Lemma 2.3, Lemma 2.4 and the proof of Lemma 2.5, implies

$$\sup_{\overline{\Omega_\delta}} \left(|\nabla u_\varepsilon|^2 + \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2} \right) \leq c(g, \Omega, \delta), \quad \Omega_\delta = \Omega \backslash \bigcup_{j=1}^d \overline{B_\delta(a_j)}. \tag{4.15}$$

Fix a $\delta > 0$ small, since

$$-\triangle u_\varepsilon^3 + \left(\frac{1}{\varepsilon^2} - \left(|\nabla u_\varepsilon|^2 + \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2} \right) \right) u_\varepsilon^3 = 0 \text{ in } \Omega. \tag{4.16}$$

We conclude from (4.15), for ε small enough, that

$$\frac{1}{\varepsilon^2} - \left(|\nabla u_\varepsilon|^2 + \frac{\left(u_\varepsilon^3\right)^2}{\varepsilon^2} \right) \geq \frac{1}{4\varepsilon^2} \text{ on } \Omega_{\frac{\delta}{2}}.$$

Since $u_\varepsilon^3 \leq 1$ and $u_\varepsilon^3|_{\partial\Omega} = 0$, by the comparison function in Lemma 4.5 and the equation (4.16) we deduce $0 \leq u_\varepsilon^3(x) \leq e^{-\frac{\delta}{16\varepsilon}}$ for $x \in \Omega_\delta$. The standard elliptic estimates and Sobolev inequality yield that $|u_\varepsilon^3|_{C^{1,\alpha}(\Omega_\delta)} \leq c(\alpha, g, \Omega, \delta) e^{-\frac{1}{c(\delta)\varepsilon}}$ for $\alpha \in (0, 1), \varepsilon$ small. On the other hand, by the first two equations we get $|u'_\varepsilon|_{C^{1,\alpha}(\Omega_\delta)} \leq c(\alpha, g, \Omega, \delta)$. An induction argument along with the Schauder estimates yields the conclusion of the theorem.

Now we want to locate the points $a_1, \ldots, a_d$ and get more information for the asymptotes. We have the following:

Theorem 4.2 *The d-tuple $a = (a_1, \ldots, a_d)$ in Theorem 4.1 minimizes the renormalized energy $W(g, \Omega, b)$, $b \in \Omega^d$, (see Section I.4 of [6] for a definition). We also have*

$$\frac{\left(u^3_{\varepsilon_i}\right)^2}{\varepsilon_i^2} \to \pi \sum_{j=1}^{d} \delta_{a_j}, \quad \frac{|\nabla u_{\varepsilon_i}|^2}{\log \frac{1}{\varepsilon_i}} \to 2\pi \sum_{j=1}^{d} \delta_{a_j}$$

in the sense of distribution, and

$$I_\varepsilon(u_\varepsilon) = \pi d \log \frac{1}{\varepsilon} + d\gamma_0 + \inf_{b \in \Omega^d} W(g, \Omega, b) + o(1),$$

where γ_0 will be explained later in Lemma 4.6. The minimizers have uniformly bounded p energy for any $1 \le p < 2$ i.e.

$$|u_\varepsilon|_{W^{1,p}(\Omega)} \le c(g, \Omega, p) \text{ for } 0 < \varepsilon \le 1.$$

For the third coordinate, we have

$$|u^3_\varepsilon|_{H^1(\Omega)} \le c(g, \Omega).$$

Denote $h(x) = \left(\frac{x}{|x|}, 0\right)$. For any $\varepsilon > 0$ and $r > 0$, let

$$I(\varepsilon, r) = \inf_{u \in H^1_h(B_r, S^2)} I_\varepsilon(u), \quad I(\varepsilon) = I(\varepsilon, 1).$$

Then $I(\varepsilon, r) = I(\frac{\varepsilon}{r})$, and we have the following:

Lemma 4.6 *For $0 < t_1 < t_2$, $I(t_1) \le \pi \log \frac{t_2}{t_1} + I(t_2)$. Especially for any $0 < t \le 1$, $I(t) \le \pi \log \frac{1}{t} + I(1)$. We also know the limit $\gamma_0 = \lim_{t \to 0^+} I(t) - \pi \log \frac{1}{t}$ exists.*

Proof For $0 < t_1 < t_2$, define

$$u(x) = \begin{cases} \left(\dfrac{x}{|x|}, 0\right) & \text{for} \quad \dfrac{1}{t_2} \le |x| \le \dfrac{1}{t_1}, \\ \text{minimizer for } I_1 \text{ on } H^1_h(B_{\frac{1}{t_2}}, S^2) & \text{for} \quad |x| < \dfrac{1}{t_2}. \end{cases}$$

Then we have

$$I(t_1) = I\left(1, \frac{1}{t_1}\right) \le I_1(u) = \pi \log \frac{t_2}{t_1} + I\left(1, \frac{1}{t_2}\right) = \pi \log \frac{t_2}{t_1} + I(t_2).$$

Hence $I(t) - \pi \log \frac{1}{t}$ is nondecreasing in t. Suppose u_ε minimizes I_ε. Then from (4.5) we have $\int_{B_1} |\nabla u_\varepsilon|^2 \ge 2\pi \log \frac{1}{\varepsilon} - c$, where c is an absolute constant. Hence $I(\varepsilon) - \pi \log \frac{1}{\varepsilon} \ge -c$.

Proof of Theorem 4.2 First we give an upper bound for $I_\varepsilon(u_\varepsilon)$ by choosing a comparison function. For $b_1, \ldots, b_d$, d different points in Ω, denote $u_b(x) = \left(\prod_{j=1}^d \frac{x - b_j}{|x - b_j|} e^{i h_b(x)}, 0\right)$, where h_b is harmonic in Ω such that $u_b|_{\partial\Omega} = g$. Choose a ρ suitably small, for each j, we know $u_b(x) = \left(\frac{x - b_j}{|x - b_j|} e^{i\Theta_j(x)}, 0\right)$ on $\overline{B_\rho(a_j)}$ and Θ_j is a harmonic function on $B_\rho(b_j)$. For $\varepsilon > 0$, $\sigma > 0$ small, define

$$v_\varepsilon(x) = \begin{cases} u_b(x) & \text{if} \quad x \in \overline{\Omega} \backslash \bigcup_{j=1}^d B_\rho(b_j), \\ \left(\dfrac{x - b_j}{|x - b_j|} e^{i\left(\Theta_j(b_j) + \frac{|x - b_j| - (1-\sigma)\rho}{\sigma\rho} (\Theta_j(x) - \Theta_j(b_j))\right)}, 0\right) & \text{if} \quad x \in B_\rho(b_j) \backslash B_{(1-\sigma)\rho}(b_j), \\ \text{minimizing } I_\varepsilon \text{ in } H^1_{\left(\frac{x - b_j}{|x - b_j|} e^{i\Theta_j(b_j)}, 0\right)}\left(B_{(1-\sigma)\rho}(b_j), S^2\right) & \text{if} \quad x \in B_{(1-\sigma)\rho}(b_j). \end{cases}$$

By the fact that

$$\frac{1}{2}\int_{\Omega\setminus\bigcup_{j=1}^{d}B_\rho(b_j)}|\nabla u_b|^2=\pi d\log\frac{1}{\rho}+W(g,\Omega,b)+o(1), \tag{4.17}$$

and Lemma 4.6 we know

$$I_\varepsilon(u_\varepsilon)\le I_\varepsilon(v_\varepsilon)\le \pi d\log\frac{1}{\varepsilon}+W(g,\Omega,b)+d\gamma_0+\alpha_1(\varepsilon,\sigma,\rho)+\alpha_2(\sigma,\rho)+\alpha_3(\sigma), \tag{4.18}$$

and

$$\lim_{\varepsilon\to0^+}\alpha_1(\varepsilon,\sigma,\rho)=0,\ \lim_{\rho\to0^+}\alpha_2(\sigma,\rho)=0,\ \lim_{\sigma\to0^+}\alpha_3(\sigma)=0.$$

The lower bound is obtained through another comparison argument. Denote

$$u_*(x)=\left(\prod_{j=1}^{d}\frac{x-a_j}{|x-a_j|}e^{ih_a(x)},0\right).$$

For $\rho>0$ small, $\sigma>0$ small, fixing j, we know $u_*(x)=\left(\frac{x-a_j}{|x-a_j|}e^{i\Theta_j(x)},0\right)$ on a disk around a_j. Define

$$w_\varepsilon(x)=\begin{cases} u_\varepsilon(x) & \text{if } x\in B_\rho(b_j),\\ \Pi\left(\dfrac{(1+\sigma)\rho-|x|}{\sigma\rho}u_\varepsilon(x)+\dfrac{|x|-\rho}{\sigma\rho}u_*(x)\right) & \text{if } x\in B_{(1+\sigma)\rho}(b_j)\setminus B_\rho(b_j),\\ \left(\dfrac{x-a_j}{|x-a_j|}e^{i\left(\Theta_j(b_j)+\frac{(1+2\sigma)\rho-|x-b_j|}{\sigma\rho}(\Theta_j(x)-\Theta_j(b_j))\right)},0\right) & \text{if } x\in B_{(1+2\sigma)\rho}(b_j)\setminus B_{(1+\sigma)\rho}(b_j).\end{cases}$$

Here $\Pi(\xi)=\frac{\xi}{|\xi|}$ for $\xi\in\mathbb{R}^3$. From Lemma 4.6 we know

$$I_\varepsilon(u_\varepsilon)\ge\frac{1}{2}\int_{\Omega_\rho}|\nabla u_\varepsilon|^2+\pi d\log\frac{\rho}{\varepsilon}+d\gamma_0-\tilde\beta_1(\varepsilon,\sigma,\rho)-\tilde\beta_2(\sigma,\rho)-\tilde\beta_3(\sigma),$$

and $\lim_{\varepsilon\to0^+}\tilde\beta_1(\varepsilon,\sigma,\rho)=0,\lim_{\rho\to0^+}\tilde\beta_2(\sigma,\rho)=0,\lim_{\sigma\to0^+}\tilde\beta_3(\sigma)=0$. Here $\Omega_\rho=\Omega\setminus\bigcup_{j=1}^{d}\overline{B_\rho(a_j)}$. From Theorem 4.1 we know

$$\begin{aligned}I_\varepsilon(u_\varepsilon)&\ge\frac{1}{2}\int_{\Omega_\rho}|\nabla u_*|^2+\pi d\log\frac{\rho}{\varepsilon}+d\gamma_0-\beta_1(\varepsilon,\sigma,\rho)-\tilde\beta_2(\sigma,\rho)-\tilde\beta_3(\sigma)\\&\ge\pi d\log\frac{1}{\varepsilon}+W(g,\Omega,a)-\beta_1(\varepsilon,\sigma,\rho)-\beta_2(\sigma,\rho)-\beta_3(\sigma),\end{aligned} \tag{4.19}$$

and $\lim_{\varepsilon\to0^+}\beta_1(\varepsilon,\sigma,\rho)=0$, $\lim_{\rho\to0^+}\beta_2(\sigma,\rho)=0$, $\lim_{\sigma\to0^+}\beta_3(\sigma)=0$. Combining (4.18) and (4.19), we get

$$W(g,\Omega,a)\le W(g,\Omega,b)+\gamma_1(\varepsilon,\sigma,\rho)+\gamma_2(\sigma,\rho)+\gamma_3(\sigma),$$

and $\lim_{\varepsilon\to0^+}\gamma_1(\varepsilon,\sigma,\rho)=0$, $\lim_{\rho\to0^+}\gamma_2(\sigma,\rho)=0$, $\lim_{\sigma\to0^+}\gamma_3(\sigma)=0$. Letting $\varepsilon\to0^+$, $\rho\to0^+$ then $\sigma\to0^+$, we get $W(g,\Omega,a)\le W(g,\Omega,b)$. This proves the first assertion in Theorem 4.2. It follows from (4.7) and (4.10) that

$$\frac{|\nabla u_{\varepsilon_i}|^2}{\log\frac{1}{\varepsilon_i}}\to2\pi\sum_{j=1}^{d}\delta_{a_j}.$$

Now choose a $\rho > 0$ suitably small; it follows from Lemma 4.4 that

$$\frac{1}{\varepsilon^2}\int_{B_\rho(a_j)}\left(u_\varepsilon^3\right)^2 + \frac{\rho}{2}\int_{\partial B_\rho(a_j)}|\partial_\nu u_\varepsilon|^2 ds = \frac{\rho}{2}\int_{\partial B_\rho(a_j)}|\partial_\tau u_\varepsilon|^2 ds + \frac{\rho}{2\varepsilon^2}\int_{\partial B_\rho(a_j)}\left(u_\varepsilon^3\right)^2 ds.$$

By Theorem 4.1 we know

$$\lim_{k\to\infty}\frac{1}{\varepsilon_k^2}\int_{B_\rho(a_j)}\left(u_{\varepsilon_k}^3\right)^2 = -\frac{\rho}{2}\int_{\partial B_\rho(a_j)}|\partial_\nu u_*|^2 ds + \frac{\rho}{2}\int_{\partial B_\rho(a_j)}|\partial_\tau u_*|^2 ds. \tag{4.20}$$

On a small disk around a_j we write $u_*(x) = \left(\frac{x-a_j}{|x-a_j|}e^{i\Theta_j(x)}, 0\right)$; under the polar coordinates we know

$$\begin{aligned}\frac{\rho}{2}\int_{\partial B_\rho(a_j)}\left(|\partial_\tau u_*|^2 - |\partial_\nu u_*|^2\right) ds &= \frac{\rho^2}{2}\left(\int_0^{2\pi}\left(\frac{1}{\rho^2}\left(1+\partial_\theta\Theta_j(\rho,\theta)\right)^2 - \left(\partial_r\Theta_j(\rho,\theta)\right)^2\right) d\theta\right)\\ &= \pi + \frac{1}{2}\int_0^{2\pi}\left(\partial_\theta\Theta_j(\rho,\theta)\right)^2 d\theta - \frac{\rho^2}{2}\int_0^{2\pi}\left(\partial_r\Theta_j(\rho,\theta)\right)^2 d\theta = \pi,\end{aligned}$$

where we have used the Pohozaev identity for Θ_j in the last step. Hence via (4.20) one has

$$\lim_{k\to\infty}\frac{1}{\varepsilon_k^2}\int_{B_\rho(a_j)}\left(u_{\varepsilon_k}^3\right)^2 = \pi.$$

The latter fact, together with Theorem 4.1, implies

$$\frac{\left(u_{\varepsilon_k}^3\right)^2}{\varepsilon_k^2} \to \pi\sum_{j=1}^d \delta_{a_j}$$

in the sense of distribution. The asymptotic formula of $I_\varepsilon(u_\varepsilon)$ follows from (4.18) and (4.19). The fact that $|u_\varepsilon^3|_{H^1(\Omega)} \le c(g,\Omega)$ follows from (4.8). The $W^{1,p}$-estimate for u_ε follows from (4.12) and the Hölder inequality.

5 Radial Solutions

In this section we shall study some special solutions of Equation (2.1). The study of these solutions would be helpful in the understanding of the dynamics of vortices. First let us look at the following boundary value problem:

$$-\triangle u = \left(|\nabla u|^2 + \frac{\left(u^3\right)^2}{\varepsilon^2}\right)u - \frac{u^3}{\varepsilon^2}e_3 \quad \text{on } B_1, \quad u(x) = \left(e^{iq\theta}, 0\right) \quad \text{for } x\in\partial B_1, \tag{5.1}$$

where $u \in C^\infty\left(\overline{B_1}, S^2\right)$, $q\in\mathbb{N}$.

Proposition 5.1 *There exists a unique $f = f_{\varepsilon,q}$ defined on $[0,1]$ such that $f(0) = 0$, $f(1) = \frac{\pi}{2}$ and $u = \left(\sin f(r)\, e^{iq\theta}, \cos f(r)\right)$ is a smooth solution to (5.1). In addition, f satisfies $0 < f(t) \le \frac{\pi}{2}$, $f'(t) > 0$ for $0 < t \le 1$.*

Proof Existence : Suppose $u = \left(\sin\rho(r)\, e^{iq\theta}, \cos\rho(r)\right)$. Then

$$I_\varepsilon(u) = \pi\int_0^1\left(r\rho'(r)^2 + \frac{q^2}{r}\sin^2\rho(r) + \frac{r\cos^2\rho(r)}{\varepsilon^2}\right) dr,$$

and we call this functional $J_\varepsilon(\rho)$. Set

$$V = \left\{\rho\backslash\rho \in H^1_{\mathrm{loc}}(0,1), \sqrt{r}\rho' \in L^2(0,1), \frac{1}{\sqrt{r}}\rho \in L^2(0,1) \text{ and } \rho(1) = \frac{\pi}{2}\right\}.$$

Since

$$\begin{aligned}|\rho(r)| &\le \frac{1}{r}\int_r^{2r}|\rho(t)|dt + \frac{1}{r}\int_r^{2r}|\rho(t)-\rho(r)|dt \\ &\le c\left(\left(\int_r^{2r}\frac{\rho(t)^2}{t}dt\right)^{\frac{1}{2}} + \left(\int_r^{2r}t\rho'(t)^2\,dt\right)^{\frac{1}{2}}\right),\end{aligned}$$

one has $\rho \in C([0,1])$ and $\rho(0) = 0$ for any $\rho \in V$. Choosing a minimizing sequence for J_ε in V, say ρ_j, then by the basic properties of trigonometric functions, we may assume $0 \le \rho_j \le \frac{\pi}{2}$. Since $\sin\rho_j \ge \frac{2}{\pi}\rho_j \ge 0$, one deduces $\sup_j \int_0^1 r\left(\rho_j'\right)^2 + \frac{1}{r}\rho_j^2 < \infty$. Assuming $\rho_j \rightharpoonup f$ in $H^1_{\mathrm{loc}}(0,1)$, then $f \in V$ and f is a minimizer. It must be smooth and it satisfies

$$f'' + \frac{f'}{r} + \left(\frac{1}{\varepsilon^2} - \frac{q^2}{r^2}\right)\frac{\sin 2f}{2} = 0. \tag{5.2}$$

This means the u corresponding to f is in $H^1\left(B_1, S^2\right)$ and satisfies (5.1) in $B_1\backslash\{0\}$, hence on the whole B_1 by a standard removable singularity theorem for such equations.

Uniqueness : Suppose f satisfies (5.2) and $f(0) = 0$, $f(1) = \frac{\pi}{2}$. We claim that $0 \le f \le \frac{\pi}{2}$. To see this we define $\varphi(t) = f(\varepsilon e^t)$ for $t \in \left(-\infty, \log\frac{1}{\varepsilon}\right]$. Then

$$\varphi''(t) = \left(q^2 - e^{2t}\right)\sin\varphi\cos\varphi,\ \varphi(-\infty) = 0,\ \varphi\left(\log\frac{1}{\varepsilon}\right) = \frac{\pi}{2}. \tag{5.3}$$

Choose a t_0 such that $|\varphi(t)| \le \frac{\pi}{4}$ for any $t \le t_0 < \log q$. Then either $\varphi(t) > 0$ for all $t \le t_0$ or $\varphi(t) < 0$ for all $t \le t_0$. In fact, if this is not the case, then there exists a $t_1 \le t_0$ such that $\varphi(t_1) = 0$. From the uniqueness theorem of o.d.e. we know $\varphi'(t_1) \ne 0$. If $\varphi'(t_1) > 0$, then $\varphi(t) < 0$ for $t < t_1$ and very close to t_1; one easily deduces $\varphi(t) < 0$ for any $t < t_1$ from (5.3). Hence $\varphi''(t) < 0$; this, combining with $\varphi'(t_1) > 0$, implies that $\varphi(-\infty) = -\infty$, which is a contradiction. Similarly, $\varphi'(t_1) < 0$ also leads to a contradiction. Let us assume $\varphi(t) > 0$ for any $t \le t_0$; then $\varphi'(t) > 0$ for $t \le t_0$. We claim $\varphi'(t) > 0$ for any $t \le \log\frac{1}{\varepsilon}$; then it follows that $0 < \varphi(t) \le \frac{\pi}{2}$. Indeed if φ' vanishes at some points, set $t_1 = \inf\left\{t\backslash t \le \log\frac{1}{\varepsilon}, \varphi'(t) = 0\right\}$. If $t_1 \le \log q$, we see $\varphi(t_1) \in \left(k\pi + \frac{\pi}{2}, k\pi + \pi\right)$ for some nonnegative integer k. By the choice of t_1 we know there exist $t_3 < t_2 < t_1$ such that $\varphi(t_2) = k\pi + \frac{\pi}{2}$, $\varphi(t_3) = 2k\pi + \pi - \varphi(t_1)$. On $[t_2, t_1]$, since $\varphi''(t) \ge \left(q^2 - e^{2t_2}\right)\sin\varphi\cos\varphi$ we get $(\varphi')^2 - \left(q^2 - e^{2t_2}\right)\sin^2\varphi|_{t_1} \ge (\varphi')^2 - \left(q^2 - e^{2t_2}\right)\sin^2\varphi|_{t_2}$. Similarly we get $(\varphi')^2 - \left(q^2 - e^{2t_2}\right)\sin^2\varphi|_{t_2} \ge (\varphi')^2 - \left(q^2 - e^{2t_2}\right)\sin^2\varphi|_{t_3}$. These two inequalities imply $\varphi'(t_3) = 0$, which is a contradiction. Hence $t_1 > \log q$, $\varphi(t_1) \in \left(k\pi, k\pi + \frac{\pi}{2}\right)$ for some nonnegative integer k. Similar arguments show that φ will be oscillating around $k\pi$ with decreasing amplitude after t_1. Hence it would not reach $\frac{\pi}{2}$ at $\log\frac{1}{\varepsilon}$, which again leads to a contradiction. If $\varphi(t) < 0$ for $t \le t_0$, then the above arguments also lead to a contradiction. Hence we dotain the claim. Setting $\phi(t) = \frac{\pi}{2} - \varphi(-t)$, $\psi = \phi^{-1}$, then ψ is a map from $\left[0, \frac{\pi}{2}\right)$

to $\left[-\log\frac{1}{\varepsilon},\infty\right)$, which satisfies

$$\psi''(x)=\left(q^2-e^{-2\psi}\right)(\psi')^3\sin x\cos x,\ \psi(0)=-\log\frac{1}{\varepsilon},\ \psi\left(\frac{\pi}{2}\right)=\infty,\ \psi'>0. \tag{5.4}$$

Given two different solutions, say f_1, f_2, we may assume $\varphi_1'\left(\log\frac{1}{\varepsilon}\right)>\varphi_2'\left(\log\frac{1}{\varepsilon}\right)>0$. Hence $\psi_2'(0)>\psi_1'(0)>0$. We observe that when $\psi_2\geq\psi_1$,

$$(\psi_2-\psi_1)''(x)\geq\left(q^2-e^{-2\psi_1}\right)\left((\psi_2')^2+\psi_2'\psi_1'+(\psi_1')^2\right)(\psi_2'-\psi_1')\sin x\cos x. \tag{5.5}$$

Thus we get $\psi_2'>\psi_1'$ on $\left[0,\frac{\pi}{2}\right)$, $\psi_2>\psi_1$ on $\left(0,\frac{\pi}{2}\right)$. Choose an $\alpha\in\left(\frac{\pi}{4},\frac{\pi}{2}\right)$ such that $t_2=\psi_2(\alpha)>\psi_1(\alpha)=t_1>-\log q$. Defining $\tilde{\phi}_1(t)=\phi_1(t-t_2+t_1)$ for $t\geq t_2$, then

$$\tilde{\phi}_1''(t)=\left(e^{-2(t+t_1-t_2)}-q^2\right)\frac{\sin 2\tilde{\phi}_1}{2}\geq\left(e^{-2t}-q^2\right)\frac{\sin 2\tilde{\phi}_1}{2}. \tag{5.6}$$

Since $\phi_2''(t)=\left(e^{-2t}-q^2\right)\sin\phi_2\cos\phi_2$, $\tilde{\phi}_1(t_2)=\phi_2(t_2)=\alpha$, $\tilde{\phi}_1'(t_2)>\phi_2'(t_2)$, we deduce $\tilde{\phi}_1(t)>\phi_2(t)$ for $t>t_2$. Hence $\tilde{\phi}_1'(t)-\phi_2'(t)\geq\tilde{\phi}_1'(t_2)-\phi_2'(t_2)$. $\tilde{\phi}_1'(t)\geq\tilde{\phi}_1'(t_2)-\phi_2'(t_2)>0$. The last statement contradicts the fact that $\tilde{\phi}_1(t)\to\frac{\pi}{2}$ as $t\to\infty$.

The main point in the above proof of uniqueness is to consider inverse functions. In fact, Equation (5.3) corresponds to a pendulum with changing gravity, 2φ corresponds to the angle between the pendulum and the upward vertical line. The reason for considering inverse functions becomes clear after looking at this model. These types of equations also appear in the study of equivariant harmonic maps. One may refer to [20], for example.

Now we go back to the equation

$$-\triangle u=\left(|\nabla u|^2+\left(u^3\right)^2\right)u-u^3e_3, \tag{5.7}$$

on the whole plane for $u\in C^\infty\left(\mathbb{R}^2,S^2\right)$. For $q\in\mathbb{N}$ fixed, if $u=\left(\sin f(r)e^{iq\theta},\cos f(r)\right)$, then Equation (5.7) changes into

$$f''+\frac{f'}{r}+\left(1-\frac{q^2}{r^2}\right)\frac{\sin 2f}{2}=0. \tag{5.8}$$

To avoid the singularity at 0, we set $\varphi(t)=f\left(e^t\right)$ for $t\in\mathbb{R}$; then,

$$\varphi''=\left(q^2-e^{2t}\right)\sin\varphi\cos\varphi. \tag{5.9}$$

Proposition 5.2 *There exists a unique $f=f_q$ defined on $[0,\infty)$ such that $f(0)=0$, $f(\infty)=\frac{\pi}{2}$ and $u=\left(\sin f(r)e^{iq\theta},\cos f(r)\right)$ is a smooth solution to (5.7). In addition, f satisfies $0<f(t)<\frac{\pi}{2}$, $f'(t)>0$ for $t>0$.*

Proof Existence : We look at (5.9) under the additional conditions $\varphi(-\infty)=0$, $\varphi(\infty)=\frac{\pi}{2}$. For any $a\in\mathbb{R}$, by Proposition 5.1 we have a φ_a defined on $(-\infty,a]$ such that $\varphi_a''=\left(q^2-e^{2t}\right)\sin\varphi_a\cos\varphi_a$, $\varphi_a(-\infty)=0$, $\varphi_a(a)=\frac{\pi}{2}$, $0\leq\varphi_a\leq\frac{\pi}{2}$. We also know $0<\varphi_a(t)<\frac{\pi}{2}$, $\varphi_a'(t)>0$ for $t<a$. From the bounds on φ_a and the equation it satisfies, we deduce that any order derivatives of φ_a are uniformly bounded for a large on any finite interval. Hence we may

find $a_j \to \infty$ such that $\varphi_{a_j} \to \varphi$ in $C^\infty(\mathbb{R})$. The limit φ satisfies (5.9) and $0 \le \varphi \le \frac{\pi}{2}$, $\varphi' \ge 0$. Next we want to show φ is nontrivial.

Claim 5.1 *There exists an $\alpha > 0$ such that $\varphi_a(\log q) \ge \alpha$ for $a \ge a_0 = \log 2q$.*

Proof of Claim 5.1 If this is not the case, then assume $\varphi_a(\log q) < \alpha$ for some $\alpha > 0$ small, and $a \ge \log 2q$. Since $\varphi_a'' \le q^2 \sin\varphi_a \cos\varphi_a$, we have $(\varphi_a')^2 - q^2\sin^2\varphi_a$ is decreasing and hence $(\varphi_a')^2 \le q^2\sin^2\varphi_a$. $\varphi_a'(\log q) \le q\sin\alpha \le q\alpha$. $\varphi_a(a_0) \le \alpha + q\alpha(a_0 - \log q) \le c(q)\alpha$. From $\varphi_a'' \le -3q^2\sin\varphi_a\cos\varphi_a$ for $t \ge a_0$ we know $(\varphi_a')^2 + 3q^2\sin^2\varphi_a$ is decreasing, so $3q^2 = (\varphi_a')^2 + 3q^2\sin^2\varphi_a|_a \le (\varphi_a')^2 + 3q^2\sin^2\varphi_a|_{a_0} \le c(q)\alpha^2$, which gives a contradiction when α is small enough.

Claim 5.2 *There exists a $\beta > 0$ such that $\varphi_a(\log q) \le \frac{\pi}{2} - \beta$ for $a \ge a_0 = \log 2q$.*

Proof of Claim 5.2 Suppose $\varphi_a(\log q) \ge \frac{\pi}{2} - \beta$ for some $a \ge a_0$ and β small. Denoting $v = \varphi_a'(\log q)$, then $\varphi_a'(t) \ge v - c(q)\beta$ for $\log q \le t \le a_0$ and hence $\beta \ge c(q)(v - c(q)\beta)^+$ which implies $v \le c(q)\beta$. $\varphi_a\left(\log\frac{q}{2}\right) \ge \frac{\pi}{2} - c(q)v \ge \frac{\pi}{2} - c(q)\beta$, but for $t \le \log\frac{q}{2}$, $\varphi_a''(t) \ge \frac{3q^2}{4}\sin\varphi_a\cos\varphi_a$. Hence $(\varphi_a')^2 - \frac{3q^2}{4}\sin^2\varphi_a$ is increasing. $0 \le (\varphi_a')^2 - \frac{3q^2}{4}\sin^2\varphi_a|_{\log\frac{q}{2}} \le v^2 - c(q)$, and this implies $1 \le c(q)\beta^2$ which can't be true when β is small enough.

Now by Claim 5.1 and Claim 5.2 we obtain $\alpha \le \varphi(\log q) \le \frac{\pi}{2} - \beta$, hence φ is not a constant function. We deduce that $\varphi' > 0$, $0 < \varphi < \frac{\pi}{2}$, $\varphi(-\infty) = 0$, $\varphi(\infty) = \frac{\pi}{2}$. It follows from the comparison function in Lemma 4.5 that φ exponentially decays at $-\infty$, hence by a direct computation we see $u = \left(\sin f(r)\, e^{iq\theta}, \cos f(r)\right)$ has finite energy on B_1. Finally by the removable singularity theorem we know u is a smooth solution of (5.7).

Uniqueness : First we observe that the arguments in the uniqueness part of Proposition 5.1 tell us $0 \le f \le \frac{\pi}{2}$ for any f which is a solution to the problem. If we have two different solutions, say f_1, f_2, then we have the corresponding φ_1, φ_2. If there exists a $t_0 \in \mathbb{R}$ such that $\varphi_1(t_0) = \varphi_2(t_0)$, we may use the proof of the uniqueness part in Proposition 5.1 to get a contradiction. In fact, one only needs to replace $\left(-\infty, \log\frac{1}{\varepsilon}\right]$ by $(-\infty, t_0]$. Without loss of generality, we assume $\varphi_2 > \varphi_1$. Choose $\alpha \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$ such that $\varphi_2^{-1}(\alpha) > \log q$, set $t_1 = \varphi_1^{-1}(\alpha) > \varphi_2^{-1}(\alpha) = t_2$. Define $\tilde\varphi_2(t) = \varphi_2(t - t_1 + t_2)$ for $t \ge t_1$. Then

$$\tilde\varphi_2''(t) = \left(q^2 - e^{2(t+t_2-t_1)}\right)\sin\tilde\varphi_2\cos\tilde\varphi_2 > \left(q^2 - e^{2t}\right)\sin\tilde\varphi_2\cos\tilde\varphi_2. \tag{5.10}$$

Since $\varphi_1''(t) = \left(q^2 - e^{2t}\right)\sin\varphi_1\cos\varphi_1$, $\tilde\varphi_2(t_1) = \varphi_1(t_1) = \alpha$, we get $\varphi_2'(t_2) = \tilde\varphi_2'(t_1) < \varphi_1'(t_1)$. In fact if $\tilde\varphi_2{}'(t_1) \ge \varphi_1'(t_1)$, then by the Taylor's expansion formula we would have $\tilde\varphi_2(t) > \varphi_1(t)$ for $t > t_1$ and very close to t_1. But when $\tilde\varphi_2 \ge \varphi_1$, we have $\tilde\varphi_2''(t) > \varphi_1''(t)$, thus one deduces that $\tilde\varphi_2(t) > \varphi_1(t)$ for $t > t_1$ and $\tilde\varphi_2'(t) - \varphi_1'(t)$ is strictly increasing. It contradicts the fact $\tilde\varphi_2'(t) \to 0$, $\varphi_1'(t) \to 0$ as $t \to \infty$. Now the fact $\varphi_2'(t_2) < \varphi_1'(t_1)$ and the arguments in the proof of the uniqueness part of Proposition 5.1 (especially (5.5), (5.6)) give us another contradiction.

We will study the stability properties of those solutions given in Proposition 5.1. For degree 1 solutions, we have:

Proposition 5.3 *If $q = 1$, then the radial solution given in Proposition* 5.1 *is strictly stable, hence a local minimizer.*

For higher-degree solutions, we have:

Proposition 5.4 *If $q \geq 2$, then for $0 < \varepsilon \leq \varepsilon(q)$, the solution given in Proposition* 5.1 *is unstable.*

Here we basically use the idea presented in [9]. Some preparations are needed. First, we would like to establish some qualitative properties of the function f in Propositions 5.1 and 5.2.

Lemma 5.1 *The function f in Proposition* 5.1 *satisfies $f \in C^{\infty}([0,1])$, $f^{(k)}(0) = 0$ for $0 \leq k \leq q-1$ and $f^{(q)}(0) > 0$.*

Proof Define a function g on $[-1,1]$ by setting $g(r) = (-1)^q f(-r)$ for $-1 \leq r \leq 0$ and $g(r) = f(r)$ for $0 \leq r \leq 1$. It follows from the proof of Proposition 5.1 that g is smooth, hence f is smooth on $[0,1]$. By using the Taylor expansion in (5.2) we get the conclusion.

Lemma 5.2 *The function f in Proposition* 5.2 *satisfies $f \in C^{\infty}([0,\infty))$, $f^{(k)}(0) = 0$ for $0 \leq k \leq q-1$, $f^{(q)}(0) > 0$ and $\frac{\pi}{2} - f$, $f^{(l)}$ exponentially decay at ∞, for any $l \in \mathbb{N}$.*

Proof The proof of the first part of Lemma 5.2 is exactly the same as in Lemma 5.1. The exponential decay property follows from Equation (5.9) and a comparison argument using the function in Lemma 4.5.

We note the method in the above proof is the same as that in the proof of Lemma 2.2 in [21], which is the first step for the shooting method.

By scaling we may assume the parameter $\varepsilon = 1$, but the domain changes from the unit ball to the ball with radius $R = \frac{1}{\varepsilon}$. To employ the arguments in [9], we use the stereographic projection Γ as defined in (4.1). Given a map $u : B_R \to S^2 \backslash \{(0,0,-1)\}$, denoting $v = \Gamma^{-1} \circ u$, then

$$I_1(u) = \frac{1}{2}\int_{B_R} \left(|\nabla u|^2 + \left(u^3\right)^2\right) dx = \int_{B_R} \frac{2}{\left(1+|v|^2\right)^2}\left(|\nabla v|^2 + \frac{\left(1-|v|^2\right)^2}{4}\right) dx. \tag{5.11}$$

We call this functional $J(v)$. Then for any $w \in H_0^1(B_R, \mathbb{C})$, we have

$$\begin{aligned} J''(v)(w) = 4\int_{B_R} \Bigg[& \frac{|\nabla w|^2}{\left(1+|v|^2\right)^2} - \frac{8\langle v,w\rangle\langle \nabla v, \nabla w\rangle}{\left(1+|v|^2\right)^3} - \frac{2|\nabla v|^2|w|^2}{\left(1+|v|^2\right)^3} \\ & - \frac{\left(1-|v|^2\right)|w|^2}{\left(1+|v|^2\right)^3} + \frac{12|\nabla v|^2\langle v,w\rangle^2}{\left(1+|v|^2\right)^4} + \frac{4\left(2-|v|^2\right)\langle v,w\rangle^2}{\left(1+|v|^2\right)^4}\Bigg] dx. \end{aligned} \tag{5.12}$$

Note that the solution in Proposition 5.1 corresponds to $v = \rho(r)e^{iq\theta}$, where $\rho = \tan\frac{f(\varepsilon r)}{2}$ and it satisfies

$$-\left(\frac{r\rho'}{\left(1+\rho^2\right)^2}\right)' - \frac{2r\rho\rho'^2}{\left(1+\rho^2\right)^3} + \left(\frac{q^2}{r} - r\right)\frac{1-\rho^2}{\left(1+\rho^2\right)^3}\rho = 0. \tag{5.13}$$

(5.13) is equivalent to

$$\rho'' = \frac{2\rho\rho'^2}{1+\rho^2} - \frac{\rho'}{r} + \left(\frac{q^2}{r^2} - 1\right)\frac{1-\rho^2}{1+\rho^2}\rho. \tag{5.14}$$

By Lemma 5.1 and Proposition 5.1, ρ has the following properties:

$$\rho \in C^{\infty}([0,R]), \quad \rho^{(k)}(0) = 0 \text{ for } 0 \leq k \leq q-1, \quad \rho^{(q)}(0) > 0, \tag{5.15}$$

and

$$0 < \rho(r) \leq 1, \quad \rho'(r) > 0 \text{ for } 0 < r \leq R. \tag{5.16}$$

We shall need the following elementary lemma later:

Lemma 5.3 *$r^2\rho^2 + r^2\rho'^2 - q^2\rho^2$ is a strictly increasing function in r. In particular, $r\rho^2 + r\rho'^2 - \frac{q^2\rho^2}{r} \geq 0$.*

Proof Denote $\psi(r) = r^2\rho^2 + r^2\rho'^2 - q^2\rho^2$. By Equation (5.14) we obtain

$$\psi'(r) = \frac{4\rho\rho'}{1+\rho^2}\psi(r) + 2r\rho^2 \geq \frac{4\rho\rho'}{1+\rho^2}\psi(r), \tag{5.17}$$

which together with $\psi(0) = 0$ implies $\psi(r) \geq 0$. Going back to Equation (5.17), we conclude that ψ is strictly increasing.

From Proposition 6.2 and Lemma 5.3, we may deduce that the radial solutions $u_{\varepsilon,q} = \left(\sin f_{\varepsilon,q}(r) e^{iq\theta}, \cos f_{\varepsilon,q}(r)\right)$, where $f_{\varepsilon,q}$ is the function in Proposition 5.1, satisfy $u_{\varepsilon,q} \to (0,0,1)$ in $C^\infty(B_1)$ as $q \to \infty$ for any fixed $\varepsilon > 0$. Moreover, $|u_{\varepsilon,q} - (0,0,1)|_{L^\infty(B_r)} = O\left(\frac{1}{q}\right)$, for any $r \in (0,1)$. Indeed, the quantity $r^2f^2 + r^2f'^2 - d^2f^2$ (see formula (21) in [9]) could be used to give a simple proof of Theorem 2, Part (b) in [9].

Now we proceed to the proof of the stability of degree 1 solutions. We follow closely the method in [9]. In contrast to the second variation formula in [9], formula (5.12) has additional first-order terms.

Proof of Proposition 5.3 Denote $Q(w) = J''(v)(w)$. To show $Q(w) \geq \alpha|w|^2_{H_0^1(B_R,\mathbb{C})}$ for some $\alpha > 0$, we only need to show $Q(w) > 0$ for any $w \neq 0$. Indeed if this is the case, set

$$\alpha = \inf\left\{Q(w) \backslash w \in H_0^1(B_R, \mathbb{C}), |w|_{H_0^1(B_R,\mathbb{C})} = 1\right\},$$

$\alpha \geq 0$. If $\alpha = 0$, then there exists $w_j \in H_0^1(B_R,\mathbb{C})$, $|w_j|_{H_0^1(B_R,\mathbb{C})} = 1$ and $Q(w_j) \to 0$. We may assume $w_j \rightharpoonup w$ in $H_0^1(B_R,\mathbb{C})$, then $Q(w) = 0$. This implies $w = 0$, $w_j \to 0$ in $L^2(B_R)$. Since $Q(w_j) \to 0$, it follows that $\int_{B_R} \frac{|\nabla w_j|^2}{(1+|v|^2)^2}dx \to 0$, and we obtain a contradiction. Thus α must be positive. Now for any $w \in H_0^1(B_R,\mathbb{C})$, we write it as

$$w = \sum_{n\in\mathbb{Z}} a_n(r) e^{in\theta}; \tag{5.18}$$

then

$$\frac{1}{2\pi}\int_{B_R} |w|^2 = \sum_{n\in\mathbb{Z}} \int_0^R r|a_n(r)|^2 dr. \tag{5.19}$$

$$\frac{1}{2\pi}\int_{B_R} |\nabla w|^2 = \sum_{n\in\mathbb{Z}} \int_0^R r\left(|a_n'(r)|^2 + \frac{n^2}{r^2}|a_n(r)|^2\right) dr. \tag{5.20}$$

$$\begin{aligned}\frac{1}{8\pi}Q(w) = \sum_{n\in\mathbb{Z}}\int_0^R \Bigg[&\frac{r}{(1+\rho^2)^2}\left(|a_n'|^2+\frac{n^2}{r^2}|a_n|^2\right)\\&-\left(\frac{2r({\rho'}^2+\frac{\rho^2}{r^2})}{(1+\rho^2)^3}+\frac{r(1-\rho^2)}{(1+\rho^2)^3}\right)|a_n|^2\\&+\left(\frac{3r\rho^2(\rho'2+\frac{\rho^2}{r^2})}{(1+\rho^2)^4}+\frac{r\rho^2(2-\rho^2)}{(1+\rho^2)^4}\right)|a_{1+n}+\bar{a}_{1-n}|^2\\&-\frac{2r\rho}{(1+\rho^2)^3}(a_{1+n}+\bar{a}_{1-n})\Bigg(\rho'(a_{1-n}'+\bar{a}_{1+n}')\\&+\frac{\rho}{r^2}((1-n)a_{1-n}+(1+n)\bar{a}_{1+n})\Bigg)\Bigg]dr. \qquad (5.21)\end{aligned}$$

By integration by parts and using Equation (5.14), noticing also that the quantities in (5.19) and (5.20) are finite, we get

$$\begin{aligned}\frac{1}{8\pi}Q\left(w\right) = \int_0^R \Bigg[&\frac{r}{\left(1+\rho^2\right)^2}\sum_{n\in\mathbb{Z}}|a_n'|^2+\frac{1}{r\left(1+\rho^2\right)^2}\sum_{n\in\mathbb{Z}}n^2|a_n|^2\\&-\left(\frac{2r\left(\rho'^2+\frac{\rho^2}{r^2}\right)}{\left(1+\rho^2\right)^3}+\frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3}\right)\sum_{n\in\mathbb{Z}}|a_n|^2\\&+\frac{2\left(r\rho^2+r\rho'^2+\frac{\rho^2}{r}+\frac{2\rho^4}{r}\right)}{\left(1+\rho^2\right)^4}\left(2|Re\left(a_1\right)|^2+\sum_{n=1}^{\infty}|a_{1+n}+\overline{a}_{1-n}|^2\right)\\&-\frac{4\rho^2}{r\left(1+\rho^2\right)^3}\left(\sum_{n\in\mathbb{Z}}n|a_n|^2+Re\left(a_1^2\right)+2\sum_{n=1}^{\infty}Re\left(a_{1-n}a_{1+n}\right)\right)\Bigg]dr\\ \geq \int_0^R\Bigg[&\frac{r}{\left(1+\rho^2\right)^2}\sum_{n\in\mathbb{Z}}|a_n'|^2+\frac{4|a_2|^2}{r\left(1+\rho^2\right)^2}+\frac{1}{r\left(1+\rho^2\right)^2}\sum_{n\neq 0,2}|a_n|^2\\&-\left(\frac{2r\left(\rho'^2+\frac{\rho^2}{r^2}\right)}{\left(1+\rho^2\right)^3}+\frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3}\right)\sum_{n\in\mathbb{Z}}|a_n|^2\\&+\frac{2\left(r\rho^2+r\rho'^2+\frac{\rho^2}{r}+\frac{2\rho^4}{r}\right)}{\left(1+\rho^2\right)^4}\sum_{n=1}^{\infty}|a_{1+n}+\overline{a}_{1-n}|^2\\&+\frac{4\left(r\rho^2+r\rho'^2-\frac{\rho^2}{r}\right)}{\left(1+\rho^2\right)^4}|Re\left(a_1\right)|^2-\frac{8\rho^2}{r\left(1+\rho^2\right)^3}\left(|a_2|^2+Re\left(a_0a_2\right)\right)\Bigg]dr\\ \geq \int_0^R\Bigg[&\frac{r}{\left(1+\rho^2\right)^2}\sum_{n\in\mathbb{Z}}|a_n'|^2+\frac{4|a_2|^2}{r\left(1+\rho^2\right)^2}+\frac{1}{r\left(1+\rho^2\right)^2}\sum_{n\neq 0,2}|a_n|^2\\&-\left(\frac{2r\left(\rho'^2+\frac{\rho^2}{r^2}\right)}{\left(1+\rho^2\right)^3}+\frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3}\right)\sum_{n\in\mathbb{Z}}|a_n|^2\\&+\frac{2\left(r\rho^2+r\rho'^2+\frac{\rho^2}{r}+\frac{2\rho^4}{r}\right)}{\left(1+\rho^2\right)^4}\sum_{n=1}^{\infty}|a_{1+n}+\overline{a}_{1-n}|^2\end{aligned}$$

$$- \frac{8\rho^2}{r\left(1+\rho^2\right)^3}\left(|a_2|^2 + Re\left(a_0 a_2\right)\right)\Bigg] dr. \tag{5.22}$$

where we have used Lemma 5.3. Set $b_0 = a_0, b_1 = \left(\sum_{n\neq 0,2} |a_n|^2\right)^{\frac{1}{2}}, b_2 = a_2$. Then

$$\begin{aligned} \frac{1}{8\pi} Q(w) \geq \int_0^R \Bigg[& \frac{r\left(|b_0'|^2 + |b_1'|^2 + |b_2'|^2\right)}{\left(1+\rho^2\right)^2} + \frac{|b_1|^2 + 4|b_2|^2}{r\left(1+\rho^2\right)^2} \\ & - \left(\frac{2r\left(\rho'^2 + \frac{\rho^2}{r^2}\right)}{\left(1+\rho^2\right)^3} + \frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3}\right)\left(|b_0|^2 + |b_1|^2 + |b_2|^2\right) \\ & + \frac{2\left(r\rho^2 + r\rho'^2 + \frac{\rho^2}{r} + \frac{2\rho^4}{r}\right)}{\left(1+\rho^2\right)^4} |\bar{b}_0 + b_2|^2 \\ & - \frac{8\rho^2}{r\left(1+\rho^2\right)^3}\left(|b_2|^2 + Re\left(b_0 b_2\right)\right)\Bigg] dr. \end{aligned} \tag{5.23}$$

Define

$$Q_1(b_1) = \int_0^R \left[\frac{r|b_1'|^2}{\left(1+\rho^2\right)^2} + \frac{|b_1|^2}{r\left(1+\rho^2\right)^2} - \left(\frac{2r\left(\rho'^2 + \frac{\rho^2}{r^2}\right)}{\left(1+\rho^2\right)^3} + \frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3}\right)|b_1|^2\right] dr. \tag{5.24}$$

We want to show for any real-valued nonzero $b_1 \in H^1_{\text{loc}}((0,R])$, if $\int_0^R \left(r|b_1'|^2 + \frac{|b_1|^2}{r}\right) dr < \infty$ and $b_1(R) = 0$, then $Q_1(b_1) > 0$. To do this we need the following:

Lemma 5.4 *$v = \rho(r) e^{i\theta}$ is the unique minimizer for J in the class*

$$\mathcal{E} = \left\{\tilde{v} \backslash \tilde{v} = g(r) e^{i\theta}, \quad \tilde{v} \in H^1(B_R, \mathbb{C}), \quad g(R) = \rho(R) = 1\right\}.$$

Proof We see there is at least one minimizer by the direct method. For any $\tilde{v}$ in $\mathcal{E}$, we have

$$J(\tilde{v}) = 4\pi \int_0^R \frac{r}{\left(1+|g|^2\right)^2}\left(|g'|^2 + \frac{|g|^2}{r^2} + \frac{\left(1-|g|^2\right)^2}{4}\right) dr \geq J\left(|g(r)| e^{i\theta}\right).$$

If $\tilde{v}$ is a minimizer, then so is $|g|e^{i\theta}$, and hence $|g|$ satisfies (5.14) with ρ replaced by $|g|$. By considering $2\arctan(|g|)$ and using the uniqueness part of Proposition 5.1 we know $|g| = \rho$. But if $g = |g|e^{i\varphi}$, then we have $|g'|^2 = |g|'^2 + |g|^2|\varphi'|^2$, hence $g = \rho$, $\tilde{v} = v$.

Lemma 5.4 yields

$$Q_1(b_1) = \frac{1}{8\pi} J''(v)\left(ib_1 e^{i\theta}\right) \geq 0.$$

If for some nonzero real b_1, $Q_1(b_1) = 0$, then

$$-\left(\frac{rb_1'}{\left(1+\rho^2\right)^2}\right)' - \frac{2r\rho'^2 b_1}{\left(1+\rho^2\right)^3} + \left(\frac{1}{r} - r\right)\frac{1-\rho^2}{\left(1+\rho^2\right)^3} b_1 = 0. \tag{5.25}$$

Note also $b_1(0) = b_1(R) = 0$. Multiplying equation (5.25) by ρ and equation (5.13) by b_1, after integration by parts we get $b_1'(R) = 0$, which implies $b_1 \equiv 0$, and we obtain a contradiction.

Hence Q_1 is strictly positive. To deal with the remaining part, we define

$$\begin{aligned} Q_2(b_0,b_2) = \int_0^R \Bigg[& \frac{r}{(1+\rho^2)^2}\left(|b_0'|^2+|b_2'|^2\right) + \frac{4|b_2|^2}{r(1+\rho^2)^2} \\ & - \left(\frac{2r\left(\rho'^2+\frac{\rho^2}{r^2}\right)}{(1+\rho^2)^3} + \frac{r(1-\rho^2)}{(1+\rho^2)^3} \right)\left(|b_0|^2+|b_2|^2\right) \\ & + \frac{2\left(r\rho^2+r\rho'^2+\frac{\rho^2}{r}+\frac{2\rho^4}{r}\right)}{(1+\rho^2)^4}|\bar{b}_0+b_2|^2 \\ & - \frac{8\rho^2}{r(1+\rho^2)^3}\left(|b_2|^2+Re(b_0b_2)\right)\Bigg]dr. \end{aligned} \tag{5.26}$$

Another equivalent form for Q_2 is the following:

$$\begin{aligned} Q_2(b_0,b_2) = \int_0^R \Bigg[& \frac{r}{(1+\rho^2)^2}\left(|b_0'|^2+|b_2'|^2\right) + \frac{4\left(r\rho^2+r\rho'^2-\frac{\rho^2}{r}\right)}{(1+\rho^2)^4}Re(b_0b_2) \\ & + \left(-\frac{2r\rho^2\rho'^2}{(1+\rho^2)^4} + \frac{r\left(\rho^4+2\rho^2-1\right)}{(1+\rho^2)^4} + \frac{2\rho^4}{r(1+\rho^2)^4} \right)|b_0|^2 \\ & + \left(-\frac{2r\rho^2\rho'^2}{(1+\rho^2)^4} + \frac{r\left(\rho^4+2\rho^2-1\right)}{(1+\rho^2)^4} + \frac{4-2\rho^4}{r(1+\rho^2)^4} \right)|b_2|^2\Bigg]dr. \end{aligned} \tag{5.27}$$

For real-valued b_0, b_2, we denote

$$\tilde{Q}_2(b_0,b_2) = Q_2(b_0,-b_2). \tag{5.28}$$

Then from Lemma 5.3, we know

$$Q_2(b_0,b_2) \geq \tilde{Q}_2(|b_0|,|b_2|).$$

Set

$$m = \inf\left\{ \tilde{Q}_2(b_0,b_2) \backslash b_0, b_2 \in H^1_{\mathrm{loc}}((0,R],\mathbb{R}), b_0(R)=b_2(R)=0, \int_0^R r\left(b_0^2+b_2^2\right)=1 \right\}.$$

If b_0, b_2 are minimizers, then we may assume $b_0 \geq b_2 \geq 0$. In fact, one simply replaces b_0 by $\max\{|b_0|,|b_2|\}$ and b_2 by $\min\{|b_0|,|b_2|\}$; the value of $\tilde{Q}_2$ decreases under this process because of (5.26), (5.27), (5.28) and Lemma 5.3. The minimizers b_0 and b_2 satisfy

$$\begin{aligned} mrb_0 = & -\left(\frac{rb_0'}{(1+\rho^2)^2}\right)' + \left(-\frac{2r\rho^2\rho'^2}{(1+\rho^2)^4} + \frac{r\left(\rho^4+2\rho^2-1\right)}{(1+\rho^2)^4} + \frac{2\rho^4}{r(1+\rho^2)^4} \right)b_0 \\ & - \frac{2\left(r\rho^2+r\rho'^2-\frac{\rho^2}{r}\right)}{(1+\rho^2)^4}b_2. \end{aligned} \tag{5.29}$$

$$mrb_2 = -\left(\frac{rb_2'}{\left(1+\rho^2\right)^2}\right)' + \left(-\frac{2r\rho^2\rho'^2}{\left(1+\rho^2\right)^4} + \frac{r\left(\rho^4+2\rho^2-1\right)}{\left(1+\rho^2\right)^4} + \frac{4-2\rho^4}{r\left(1+\rho^2\right)^4}\right) b_2 - \frac{2\left(r\rho^2 + r\rho'^2 - \frac{\rho^2}{r}\right)}{\left(1+\rho^2\right)^4} b_0. \tag{5.30}$$

If we define $A = \frac{b_0-b_2}{2} \geq 0$, $B = \frac{b_0+b_2}{2} \geq 0$, then

$$mrA = -\left(\frac{rA'}{\left(1+\rho^2\right)^2}\right)' + \left(\frac{2r\left(1-\rho^2\right)\rho'^2}{\left(1+\rho^2\right)^4} + \frac{r\left(\rho^4+4\rho^2-1\right)}{\left(1+\rho^2\right)^4} + \frac{2\left(1-\rho^2\right)}{r\left(1+\rho^2\right)^4}\right) A + \frac{2\left(\rho^2-1\right)}{r\left(1+\rho^2\right)^3} B. \tag{5.31}$$

$$mrB = -\left(\frac{rB'}{\left(1+\rho^2\right)^2}\right)' + \left(-\frac{2r\rho'^2}{\left(1+\rho^2\right)^3} - \frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3} + \frac{2}{r\left(1+\rho^2\right)^3}\right) B + \frac{2\left(\rho^2-1\right)}{r\left(1+\rho^2\right)^3} A. \tag{5.32}$$

Denoting $A_1 = \rho' > 0$, $B_1 = \frac{\rho}{r} > 0$, then

$$0 = -\left(\frac{rA_1'}{\left(1+\rho^2\right)^2}\right)' + \left(\frac{2r\left(1-\rho^2\right)\rho'^2}{\left(1+\rho^2\right)^4} + \frac{r\left(\rho^4+4\rho^2-1\right)}{\left(1+\rho^2\right)^4} + \frac{2\left(1-\rho^2\right)}{r\left(1+\rho^2\right)^4}\right) A_1 + \frac{2\left(\rho^2-1\right)}{r\left(1+\rho^2\right)^3} B_1. \tag{5.33}$$

$$0 = -\left(\frac{rB_1'}{\left(1+\rho^2\right)^2}\right)' + \left(-\frac{2r\rho'^2}{\left(1+\rho^2\right)^3} - \frac{r\left(1-\rho^2\right)}{\left(1+\rho^2\right)^3} + \frac{2}{r\left(1+\rho^2\right)^3}\right) B_1 + \frac{2\left(\rho^2-1\right)}{r\left(1+\rho^2\right)^3} A_1. \tag{5.34}$$

Multiplying (5.31) and (5.32) by A_1 and B_1, respectively, and multiplying (5.33) and (5.34) by A and B, respectively, then after integrating by parts, we get $-m\int_0^R r\left(AA_1 + BB_1\right) dr = \frac{R}{4}\left(A_1(R)\, A'(R) + B_1(R)\, B'(R)\right)$.

Observing A and B reach a minimum at R, we get $m \geq 0$. If $m = 0$, then $A'(R) = B'(R) = 0$. The latter fact together with (5.31), (5.32) shows $A = B = 0$. Hence $b_0 = b_2 = 0$, we obtain a contradiction. Grouping the above two results together, also using (5.23) we get $Q(w) > 0$ for any $w \neq 0$. Finally to prove u is a local minimizer, we only need to observe $I_1(\tilde{u}) = I_1\left(\tilde{u}^1, \tilde{u}^2, |\tilde{u}^3|\right)$ and the continuity of the map which sends $\tilde{u}$ to $\left(\tilde{u}^1, \tilde{u}^2, |\tilde{u}^3|\right)$.

Proof of Proposition 5.4 Suppose $q \geq 2$. For each $R > 0$, we have

$$u_R = \left(\sin f_{\frac{1}{R},q}\left(\frac{r}{R}\right) e^{iq\theta}, \cos f_{\frac{1}{R},q}\left(\frac{r}{R}\right)\right), \quad u = \left(\sin f_q(r)\, e^{iq\theta}, \cos f_q(r)\right),$$

$$v_R = \Gamma^{-1} \circ u_R = \rho_R(r)\, e^{iq\theta}, \quad v = \Gamma^{-1} \circ u = \rho(r)\, e^{iq\theta}.$$

Here Γ is the stereographic projection defined in (4.1) and $f_{\frac{1}{R},q}$, f_q are the solutions in Propositions 5.1 and 5.2 respectively. From Proposition 6.2 we know for any $R_0 > 0$, $\sup_{|x|\leq R_0, R\geq R_0+1} |\nabla\, u_R\,(x)\,| \leq c$, where c is an absolute constant. Following the proof of Proposition 5.2 one gets $u_R \to u$ in $C^\infty\left(\mathbb{R}^2, S^2\right)$. Hence for any $w \in H^1\left(\mathbb{R}^2, \mathbb{C}\right)$ with compact support, $J''\left(v_R\right)(w) \to J''(v)(w)$ as $R \to \infty$. To establish the instability, it suffices to find a $w \in H^1\left(\mathbb{R}^2, \mathbb{C}\right)$ such that $J''(v)(w) < 0$. The idea for choosing such a w is from [22]. We take

$$w = \left(\frac{\rho'}{r} + \frac{q\rho}{r^2}\right) e^{i(q-2)\theta} + \left(\frac{\rho'}{r} - \frac{q\rho}{r^2}\right) e^{i(q+2)\theta};$$

by Lemma 5.2 we see $\rho = \tan\frac{f_q}{2}$ satisfies (5.14) and

$$\rho \in C^\infty\left([0,\infty)\right), \quad \rho^{(k)}(0) = 0 \text{ for } 0 \leq k \leq q-1, \quad \rho^{(q)}(0) > 0. \tag{5.35}$$

Moreover,

$$0 < \rho(r) < 1,\ \rho'(r) > 0 \text{ for } r > 0,\ 1-\rho, \rho^{(l)} \text{ exponentially decay at } \infty,\ l \in \mathbb{N}. \tag{5.36}$$

From these we conclude that $w \in H^1\left(\mathbb{R}^2, \mathbb{C}\right)$. Plugging v and w into (5.12), using (5.14), (5.35) and (5.36) we get

$$\frac{1}{16\pi} J''(v)(w) = -\int_0^\infty \frac{2\left(\rho - \rho^3\right)\rho'}{r^2\left(1+\rho^2\right)^3} dr < 0. \tag{5.37}$$

This completes the proof.

6 Quantization

In this section we shall study the solution of Equation (5.7) on the whole plane for $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ satisfying certain growth conditions.

Proposition 6.1 *Suppose $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ satisfies (5.7) on $\mathbb{R}^2$, $u^3 \to 0$ as $|x| \to \infty$ and there exists $c > 0$ such that*

$$\int_{B_r} \left(|\nabla u|^2 + \left(u^3\right)^2\right) \leq c \log r \quad \text{for } r \geq 2. \tag{6.1}$$

Then $\int_{\mathbb{R}^2} \left(u^3\right)^2 = \pi d^2$, where d is the degree of $\frac{u'}{|u'|}$ at ∞, $u' = \left(u^1, u^2\right)$. Moreover,

$$|D^k u^3(x)\,| \leq c(k,u)\, e^{-c|x|},\ c > 0,\ c(k,u) > 0, \quad \text{for any } k \geq 0,$$

$$|D^k u(x)\,| \leq \frac{c(k,u)}{|x|^k} \text{ for } k \geq 1.$$

If we write $u^1 + iu^2 = \rho e^{i(d\theta+\psi)}$ outside ball B_{R_0}, then $|\nabla\psi(x)\,| = O\left(\frac{1}{|x|^2}\right)$.

We note that those radial solutions given in Proposition 5.2 satisfy all conditions in Proposition 6.1 and $\int_{\mathbb{R}^2}\left(u^3\right)^2 = \pi q^2$.

Proof of Proposition 6.1 We start with the following:

Claim 6.1 $\int_{\mathbb{R}^2} \left(u^3\right)^2 dx < \infty$.

Proof of Claim 6.1 Denote $\phi(r) = \int_{\partial B_r} \left[|\nabla u|^2 + \left(u^3\right)^2\right] ds$. For $r \geq 4$, we have

$$\begin{aligned} c \log r &\geq \int_{B_r} \left[|\nabla u|^2 + \left(u^3\right)^2\right] dx = \int_0^r \phi(\rho) d\rho \\ &\geq \int_{\sqrt{r}}^r \frac{\phi(\rho)\rho}{\rho} d\rho \geq \frac{1}{2} \log r \inf_{\sqrt{r} \leq \rho \leq r} \rho\phi(\rho), \end{aligned}$$

hence $\inf_{\sqrt{r} \leq \rho \leq r} \rho\phi(\rho) \leq 2c$. We may find a sequence $r_j \to \infty$ such that $r_j \phi(r_j) \leq 2c$. From Lemma 4.4 we know

$$\int_{B_r} \left(u^3\right)^2 + \frac{r}{2} \int_{\partial B_r} |\partial_\nu u|^2 ds = \frac{r}{2} \int_{\partial B_r} |\partial_\tau u|^2 ds + \frac{r}{2} \int_{\partial B_r} \left(u^3\right)^2 ds. \tag{6.2}$$

Hence

$$\int_{B_{r_j}} \left(u^3\right)^2 \leq \frac{r_j}{2} \int_{\partial B_{r_j}} \left(|\partial_\tau u|^2 + \left(u^3\right)^2\right) ds \leq \frac{r_j}{2} \phi\left(r_j\right),$$

which implies $\int_{\mathbb{R}^2} \left(u^3\right)^2 dx \leq c < \infty$.

By assumptions, we may choose $R_0 > 0$ such that $|u^3(x)| \leq \frac{1}{2}$ on $\mathbb{R}^2 \backslash B_{R_0}$, then $u' = u^1 + iu^2 = \rho e^{i\varphi}$, where $\rho = |u'| \geq \frac{\sqrt{3}}{2}$, $\varphi = d\theta + \psi$, d being the degree of $u'/|u'|$ at ∞, ψ being a single-valued smooth function on $\mathbb{R}^2 \backslash B_{R_0}$. Computation shows $\operatorname{div}\left(\rho^2 \nabla\varphi\right) = 0$.

Claim 6.2 $\int_{\mathbb{R}^2 \backslash B_{R_0}} |\nabla\psi|^2 < \infty$.

Proof of Claim 6.2 Denote $A_R = B_R \backslash \overline{B_{R_0}}$, $\psi_R = \frac{1}{2\pi R} \int_{\partial B_R} \psi$. Then

$$\begin{aligned} \int_{A_R} \rho^2 \left(d\nabla\theta + \nabla\psi\right) \cdot \nabla\psi &= \int_{A_R} \operatorname{div}\left(\psi\rho^2\nabla\varphi\right) \\ &= \left(\int_{\partial B_R} - \int_{\partial B_{R_0}}\right) \rho^2 \frac{\partial\varphi}{\partial\nu} \psi ds = \int_{\partial B_R} \rho^2 \frac{\partial\psi}{\partial\nu} \left(\psi - \psi_R\right) ds - c(u). \end{aligned}$$

Here we use the fact that $\int_{\partial B_R} \rho^2 \frac{\partial\psi}{\partial\nu} ds = 0$, which results from

$$\int_{\partial B_R} \rho^2 \frac{\partial\psi}{\partial\nu} ds = \int_{\partial B_R} u' \times \partial_\nu u' ds = \int_{B_R} \operatorname{div}\left(u' \times \partial_1 u', u' \times \partial_2 u'\right) dx = 0.$$

Since $\int_{\partial B_r} \nabla\theta \cdot \nabla\psi ds = 0$, we get

$$\int_{A_R} \rho^2 |\nabla\psi|^2 \leq \int_{\partial B_R} \left|\frac{\partial\psi}{\partial\nu}\right| |\psi - \psi_R| ds + \int_{A_R} \left(1 - \rho^2\right) \frac{|d|}{r} |\nabla\psi| + c(u).$$

By the Hölder and Poincaré inequalities we have

$$\int_{\partial B_R} \left|\frac{\partial\psi}{\partial\nu}\right| |\psi - \psi_R| ds \leq \frac{R}{2} \int_{\partial B_R} |\nabla\psi|^2 ds,$$

and

$$\int_{A_R} \left(1 - \rho^2\right) \frac{|d|}{r} |\nabla\psi| \leq \frac{|d|}{R_0} \left(\int_{A_R} \left(u^3\right)^4\right)^{\frac{1}{2}} \left(\int_{A_R} |\nabla\psi|^2\right)^{\frac{1}{2}} \leq c(u) \left(\int_{A_R} |\nabla\psi|^2\right)^{\frac{1}{2}}.$$

It follows that

$$\int_{A_R} |\nabla\psi|^2 \le c_0 R \int_{\partial B_R} |\nabla\psi|^2 ds + c(u).$$

Since $\int_{A_R} |\nabla\psi|^2 \le c(u)\log R$, by choosing a generic R we get $\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\psi|^2 \le c(u) < \infty$.

Claim 6.3 $\int_{\mathbb{R}^2} |\nabla u^3|^2 < \infty$.

Proof of Claim 6.3 In fact, multiplying the third component's equation by u^3 and integrating by parts we get

$$\int_{A_R} |\nabla u^3|^2 + \left(u^3\right)^2 = \int_{A_R} \left(|\nabla u|^2 + \left(u^3\right)^2\right)\left(u^3\right)^2 + \int_{\partial B_R} u^3 \frac{\partial u^3}{\partial\nu} ds - \int_{\partial B_{R_0}} u^3 \frac{\partial u^3}{\partial\nu} ds.$$

Hence

$$\begin{aligned}\int_{A_R} \frac{1-2\left(u^3\right)^2}{1-\left(u^3\right)^2}|\nabla u^3|^2 + \left(u^3\right)^2 &= \int_{A_R} \left(u^3\right)^2 \rho^2 |\nabla\varphi|^2 + \int_{\partial B_R} u^3 \frac{\partial u^3}{\partial\nu} ds - \int_{\partial B_{R_0}} u^3 \frac{\partial u^3}{\partial\nu} ds \\ &\le \frac{2d^2}{R_0^2}\int_{\mathbb{R}^2} \left(u^3\right)^2 + 2\int_{A_R} |\nabla\psi|^2 + \int_{\partial B_R} u^3 \frac{\partial u^3}{\partial\nu} ds - \int_{\partial B_{R_0}} u^3 \frac{\partial u^3}{\partial\nu} ds \\ &\le c(u) + \left(R\int_{\partial B_R} |\nabla u^3|^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Choose $R_j \to \infty$ such that $R_j \int_{\partial B_{R_j}} |\nabla u^3|^2 ds \le 2c$ is independent of j, then $\int_{\mathbb{R}^2} |\nabla u^3|^2 \le c(u)$.

For ρ we have

$$\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho|^2 = \int_{\mathbb{R}^2\setminus B_{R_0}} \frac{\left(u^3\right)^2 |\nabla u^3|^2}{1-\left(u^3\right)^2} < \infty.$$

One observes that

$$|\partial_\tau u'|^2 = |\partial_\tau\rho|^2 + \frac{d^2\rho^2}{r^2} + \frac{2\rho^2 d\partial_\tau\psi}{r} + \rho^2\left(\partial_\tau\psi\right)^2, \quad |\partial_\nu u'|^2 = \left(\partial_\nu\rho\right)^2 + \rho^2\left(\partial_\nu\psi\right)^2.$$

Also by (6.2) one sets

$$\int_{B_r} \left(u^3\right)^2 = \pi d^2 + \mathcal{R},$$

where

$$|\mathcal{R}| \le c(u)\left(r\int_{\partial B_r}\left(|\nabla\rho|^2 + |\nabla\psi|^2 + |\nabla u^3|^2 + \left(u^3\right)^2\right) ds + \left(r\int_{\partial B_r} |\nabla\psi|^2 ds\right)^{\frac{1}{2}}\right).$$

Using the fact that $\int_{\mathbb{R}^2\setminus B_{R_0}} |\nabla\rho|^2 + |\nabla\psi|^2 + |\nabla u^3|^2 + \left(u^3\right)^2 < \infty$, we may find $r_j \to \infty$ such that

$$r_j \int_{\partial B_{r_j}} \left(|\nabla\rho|^2 + |\nabla\psi|^2 + |\nabla u^3|^2 + \left(u^3\right)^2\right) ds \to 0.$$

Hence $\int_{\mathbb{R}^2} \left(u^3\right)^2 dx = \pi d^2$. Next we look at $e(u) = \frac{1}{2}\left[|\nabla u|^2 + \left(u^3\right)^2\right]$. Fixing an x, letting $R = |x|$, for $\delta > 0$ small, $R > 2R_0$, we have

$$\begin{aligned}\int_{B_{(1+\delta)R}\setminus B_{(1-\delta)R}} |\nabla u|^2 &= \int_{B_{(1+\delta)R}\setminus B_{(1-\delta)R}} |\nabla u^3|^2 + |\nabla\rho|^2 + \rho^2 |\nabla\left(d\theta + \psi\right)|^2 \\ &\le 4\pi d^2 \log\frac{1+\delta}{1-\delta} + \int_{B_{(1+\delta)R}\setminus B_{(1-\delta)R}} |\nabla u^3|^2 + |\nabla\rho|^2 + 2|\nabla\psi|^2.\end{aligned}$$

Since $\int_{\mathbb{R}^2\backslash B_{R_0}} |\nabla u^3|^2 + |\nabla\rho|^2 + 2|\nabla\psi|^2 + \left(u^3\right)^2$ is finite, one can make $\int_{B_{(1+\delta)R}\backslash B_{(1-\delta)R}} e(u)$ arbitrarily small if one takes R sufficiently large and δ sufficiently small. Using Lemma 2.3 and Lemma 2.4 on $B_{\delta R}(x)$, we get $e(u)(x) \leq \frac{c(u)}{|x|^2}$. Hence $|\nabla u(x)| \leq \frac{c(u)}{|x|}$. Next we let R_0 be large enough such that for $|x| \geq R_0$, $|\nabla u(x)|^2 + \left(u^3(x)\right)^2 \leq \frac{3}{4}$. Then from the third component's equation and the comparison function in Lemma 4.5 we get that $u^3(x)$ exponentially decays at infinity. The estimate of $|\nabla u^3(x)|$ follows from the standard elliptic estimates. By scaling, the Schauder theory, and induction we get the estimates for $|D^k u|$ and $|D^k u^3|$. We note that ψ satisfies

$$\triangle\psi = -\frac{2\nabla\rho\cdot\nabla\varphi}{\rho} = f \quad \text{ on } \mathbb{R}^2\backslash B_{R_0}, \tag{6.3}$$

where $|f(x)| \leq c(u)\, e^{-c|x|}$. Consider the Kelvin transformation of ψ defined by $\tilde{\psi}(x) = \psi\left(\frac{x}{|x|^2}\right)$ for $x \in B_{\frac{1}{R_0}}$; then

$$\triangle\tilde{\psi} = \tilde{f} \text{ on } B_{\frac{1}{R_0}}\backslash\{0\},\ \tilde{f}(x) = \frac{1}{|x|^4} f\left(\frac{x}{|x|^2}\right),\ \int_{B_{\frac{1}{R_0}}} |\nabla\tilde{\psi}|^2 = \int_{\mathbb{R}^2\backslash B_{R_0}} |\nabla\psi|^2 < \infty. \tag{6.4}$$

We also have $|\nabla\tilde{\psi}(x)| = \frac{1}{|x|^2}|\nabla\psi(\frac{x}{|x|^2})|$. Since $|\tilde{f}(x)| \leq \frac{c(u)}{|x|^4}e^{-\frac{c}{|x|}} \to 0$ as $x \to 0$, from the removable singularity theorem we get $\tilde{\psi} \in C^1(B_1)$ and $|\nabla\psi(x)| = O(\frac{1}{|x|^2})$.

Remark 6.1 Suppose we have a $u \in C^\infty\left(\mathbb{R}^2, S^2\right)$ satisfying (5.7) and $\int_{\mathbb{R}^2} |\nabla u|^2 < \infty$. Then u must be a constant which is equal to $(0,0,1)$ or $(0,0,-1)$ or a point in S^1. In fact, we may get an L^∞ bound of ∇u by Lemma 2.3 and Lemma 2.4. Multiplying the equation of the third component by u^3 and integrating by parts we get

$$\int_{B_R} |\nabla u^3|^2 + \left(u^3\right)^2 = \int_{B_R}\left(|\nabla u|^2 + \left(u^3\right)^2\right)\left(u^3\right)^2 + \int_{\partial B_R} u^3\partial_\nu u^3 ds.$$

By choosing a sequence of generic R's which goes to ∞, one can obtain

$$\int_{\mathbb{R}^2}\left(u^3\right)^2\left(1-\left(u^3\right)^2\right)dx < \infty.$$

This last statement and the gradient bounds implies either $\int_{\mathbb{R}^2}\left(u^3\right)^2 dx$ or $\int_{\mathbb{R}^2}(1-(u^3)^2)dx$ must be finite; now by using (6.2) for a generic r, we get the conclusion.

The next proposition shows any smooth solution of (5.7) with an image in the open upper half sphere satisfies the gradient estimate.

Proposition 6.2 *Suppose* $u \in C^\infty\left(\overline{B_1}, S^2\right)$ *satisfies* (5.7) *on* B_1, $u^3 \geq 0$. *Then either* $u^3 \equiv 0$ *or* $|\nabla u(x)| \leq \frac{c}{1-|x|}$ *for* $x \in B_1$, *where* c *is an absolute constant.*

Proof Suppose the proposition is false; then we might find a sequence $u_j \in C^\infty\left(\overline{B_1}, S^2\right)$ satisfying (5.7) on B_1, $u_j^3 \geq 0$, u_j^3 not identically zero and

$$K_j = \sup_{x\in\overline{B_1}}(1-|x|)\,|\nabla u_j(x)| \to \infty.$$

By Harnack's inequality one has $u_j^3 > 0$ in B_1, hence the first eigenvalue of the operator $-\triangle + (1-(|\nabla u_j|^2 + (u_j^3)^2))$ on B_r with the Dirichlet boundary condition is positive, for any

$0<r<1$. This implies that for any $\psi \in C_c^{\infty}(B_1)$,

$$\int_{B_1} |\nabla \psi|^2 + \psi^2 \geq \int_{B_1} \left(|\nabla u_j|^2 + \left(u_j^3\right)^2 \right) \psi^2. \tag{6.5}$$

Choose $x_j \in B_1$ such that $K_j = (1-|x_j|)\,|\nabla u_j(x_j)|$, put $\sigma_j = 1-|x_j|$, define $v_j(x) = u_j\left(x_j + \frac{\sigma_j}{K_j} x\right)$ for $x \in B_{K_j}$. Then

$$-\triangle v_j = \left(|\nabla v_j|^2 + \frac{\sigma_j^2 \left(v_j^3\right)^2}{K_j^2} \right) v_j - \frac{\sigma_j^2 v_j^3}{K_j^2} e_3 \quad \text{ on } B_{K_j},$$

$$|\nabla v_j(x)| \leq \frac{1}{1-\frac{|x|}{K_j}}, \quad |\nabla v_j(0)| = 1, \quad v_j^3 \geq 0.$$

Hence $|v_j|_{C^{1,\alpha}\left(\overline{B_r}\right)} \leq c(\alpha, r)$ for $0<\alpha<1$, $r>0$. After passing to a subsequence we may assume $v_j \to v$ in $C^{\infty}\left(\mathbb{R}^2\right)$. Then $v \in C^{\infty}\left(\mathbb{R}^2, S^2\right)$ and

$$-\triangle v = |\nabla v|^2 v \text{ on } \mathbb{R}^2, \quad |\nabla v(x)| \leq 1, \quad v^3(x) \geq 0, \quad |\nabla v(0)| = 1.$$

By Lemma 3.2, $v(x) = (e^{i\left(c_0 + c_1 x^1 + c_2 x^2\right)}, 0)$, c_0, c_1, c_2 being real constants with $c_1^2 + c_2^2 = 1$. On the other hand, by choosing a suitable ψ in (6.5) we have, for any $R>0$,

$$\int_{B_R} |\nabla v_j|^2 = \int_{B_{\frac{\sigma_j}{K_j} R}(x_j)} |\nabla u_j|^2 \leq c + c \frac{\sigma_j^2}{K_j^2} R^2,$$

where c is an absolute constant. Letting $j \to \infty$, we get $\int_{B_R} |\nabla v|^2 \leq c$, hence $\pi R^2 \leq c$ for any R, which is a contradiction.

For the gradient estimate up to the boundary, we have the following proposition which in fact deals with more general solutions than Theorem 3.1:

Proposition 6.3 *Suppose $\Omega \subset \mathbb{R}^2$ is a bounded open domain with smooth boundary and $g: \partial\Omega \to S^1$ is a smooth map. If $u_{\varepsilon} \in C^{\infty}\left(\overline{\Omega}, S^2\right)$ satisfies (2.1) and $u_{\varepsilon}^3 > 0$ in Ω, then for $0 < \varepsilon \leq \varepsilon_*(g, \Omega)$, we have $|\nabla u_{\varepsilon}(x)| \leq \frac{c(g,\Omega)}{\varepsilon}$ for $x \in \overline{\Omega}$.*

Proof Suppose to the contrary that there are a sequence $\varepsilon_j \to 0$, and a sequence $u_j = u_{\varepsilon_j}$, solutions of (2.1) such that $u_j^3 > 0$ in Ω and that

$$K_j = \varepsilon_j \sup_{x \in \overline{\Omega}} |\nabla u_j(x)| \to \infty.$$

Then the eigenvalue argument in Proposition 6.2 implies, for any $\psi \in C_c^{\infty}(\Omega)$,

$$\int_{\Omega} |\nabla \psi|^2 + \frac{\psi^2}{\varepsilon_j^2} \geq \int_{\Omega} \left(|\nabla u_j|^2 + \frac{\left(u_j^3\right)^2}{\varepsilon_j^2} \right) \psi^2. \tag{6.6}$$

Choose $x_j \in \overline{\Omega}$ such that $K_j = \varepsilon_j |\nabla u_j(x_j)|$, define $\Omega_j = \frac{K_j}{\varepsilon_j}(\Omega - x_j)$, $v_j(x) = u_j\left(x_j + \frac{\varepsilon_j}{K_j} x\right)$ for $x \in \overline{\Omega_j}$. Then

$$-\triangle v_j = \left(|\nabla v_j|^2 + \frac{\left(v_j^3\right)^2}{K_j^2} \right) v_j + \frac{v_j^3}{K_j^2} e_3 \ \text{ on } \Omega_j, \quad |\nabla v_j(x)| \leq 1, \quad |\nabla v_j(0)| = 1, \quad v_j^3 \geq 0.$$

If $\Omega_j \to \mathbb{R}^2$, then $|v_j|_{C^{1,\alpha}(\overline{B_r})} \le c(\alpha, r)$ for $0 < \alpha < 1$, $r > 0$. Hence after passing to a subsequence we may assume $v_j \to v$ in $C^\infty(\mathbb{R}^2)$. Then $v \in C^\infty(\mathbb{R}^2, S^2)$ and

$$-\triangle v = |\nabla v|^2 v \text{ on } \mathbb{R}^2, \quad |\nabla v(x)| \le 1, \quad |\nabla v(0)| = 1, \quad v^3(x) \ge 0.$$

We deduce from Lemma 3.2 that $v(x) = \left(e^{i\left(c_0 + c_1 x^1 + c_2 x^2\right)}, 0\right)$, c_0, c_1, c_2 being real constants such that $c_1^2 + c_2^2 = 1$. On the other hand, by choosing a suitable ψ in (6.6) we have, for any $R > 0$,

$$\int_{B_R} |\nabla v_j|^2 = \int_{B_{\frac{\varepsilon_j}{K_j} R}(x_j)} |\nabla u_j|^2 \le c + c\frac{R^2}{K_j^2}.$$

Letting $j \to \infty$, we get $\int_{B_R} |\nabla v|^2 \le c$, hence we obtain a contradiction. If $\Omega_j \to H$, H is a half plane, after rotation we may assume $H = \{x \backslash x \in \mathbb{R}^2, x^2 > -a\}$ for some $a \ge 0$, then $v_j \to v$ in $C^\infty(\overline{H})$. $v \in C^\infty(\overline{H}, S^2)$ and $v|_{\partial H}$ is a constant in S^1. Since v is a smooth harmonic map from $\overline{H}$ to S^2, $v^3 \ge 0$, $|\nabla v(x)| \le 1$ and $|\nabla v(0)| = 1$, and due to (6.6) we get, for any ball $B_R(x_0)$ such that $B_{2R}(x_0) \subset H$, we have the estimate

$$\int_{B_R(x_0)} |\nabla v|^2 \le c. \tag{6.7}$$

Here c is an absolute constant. The Hopf function $\varphi = |\partial_1 v|^2 - |\partial_2 v|^2 - 2i(\partial_1 v \cdot \partial_2 v)$ is holomorphic and bounded, $\mathrm{Im}(\varphi) = 0$ on ∂H, hence $\varphi \equiv$ const. But from (6.7) we know $\int_{B_R(x_0)} |\varphi| \le c$ for any $B_{2r}(x_0) \subset H$, which implies $\varphi \equiv 0$. Hence $\partial_2 v \equiv 0$ on ∂H. $v \equiv$ const by the uniqueness of the Cauchy problem. This contradicts the fact $|\nabla v(0)| = 1$.

Proposition 6.4 *Suppose $u \in C^\infty(\mathbb{R}^2, S^2)$ satisfies (5.7) on the whole plane, $u^3 \ge 0$, $\liminf_{|x|\to\infty} |\nabla u(x)| = 0$, $\int_{\mathbb{R}^2} (u^3)^2 dx < \infty$. Then either $u^3 \equiv 0$ or*

$$|u^3(x)| \le c(u)e^{-\frac{|x|}{16}}, \ |\nabla u^3(x)| \le c(u)e^{-\frac{|x|}{16}}, \ |\nabla u(x)| \le \frac{c(u)}{|x|}, \ \int_{\mathbb{R}^2} (u^3)^2 dx = \pi d^2,$$

where d is the degree of $\frac{u'}{|u'|}$ at ∞ with, $u' = (u^1, u^2)$.

Proof If u^3 is not identically zero, from Harnack's inequality we know $u^3 > 0$. Proposition 6.2 tells us $|\nabla u(x)| \le c$. This and the fact that $\int_{\mathbb{R}^2} (u^3)^2 < \infty$ imply $u^3(x) \to 0$ as $|x| \to \infty$. Hence one has $|\nabla u^3(x)| \to 0$ by the elliptic estimates.

Claim 6.4 $\int_{\mathbb{R}^2} |\nabla u^3(x)|^2 dx < \infty$.

Proof of Claim 6.4 In fact, from the third component's equation we get

$$\begin{aligned}\int_{B_r} |\nabla u^3|^2 + (u^3)^2 &= \int_{B_r} (u^3)^2 \left(|\nabla u|^2 + (u^3)^2\right) + \int_{\partial B_r} u^3 \frac{\partial u^3}{\partial \nu} ds \\ &\le c\int_{\mathbb{R}^2} (u^3)^2 + c\left(r\int_{\partial B_r} (u^3)^2 ds\right)^{\frac{1}{2}}.\end{aligned}$$

Choosing a generic r, we get $\int_{\mathbb{R}^2} |\nabla u^3|^2 \le c\int_{\mathbb{R}^2} (u^3)^2 < \infty$.

Consider the Hopf function $\varphi = (|\partial_1 u|^2 - |\partial_2 u|^2) - 2i(\partial_1 u \cdot \partial_2 u)$. Then $\frac{\partial \varphi}{\partial \bar{z}} = \frac{\partial (u^3)^2}{\partial z}$.

Claim 6.5 $\left|\frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}(x)\right| \to 0$ *as* $|x| \to \infty$.

Proof of Claim 6.5 Fixing a positive number r_0, then for $|x| > r_0$, we have

$$\begin{aligned}\left|\frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}(x)\right| &\le \frac{1}{\pi r_0}\int_{\mathbb{R}^2}\left|\frac{\partial\left(u^3\right)^2}{\partial z}\right| + cr_0 \sup_{y\in B_{r_0}} \left|\left(u^3\nabla u^3\right)(x-y)\right| \\ &\le \frac{c(u)}{r_0} + cr_0 \sup_{y\in B_{r_0}} \left|\left(u^3\nabla u^3\right)(x-y)\right|.\end{aligned}$$

Letting $|x| \to \infty$ then $r_0 \to \infty$, we get Claim 6.5.

Now $\varphi - \frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}$ is holomorphic and bounded, so it must be a constant. From the condition $\liminf_{|x|\to\infty} |\nabla u(x)| = 0$ we know $\varphi = \frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}$ and $\varphi \to 0$ as $|x| \to \infty$.

Claim 6.6 $|\nabla u(x)| \to 0$ *as* $|x| \to \infty$.

Proof of Claim 6.6 Choose any $x_j \to \infty$, let $v_j(x) = u(x_j + x)$. Then after passing to the subsequence $v_j \to v$ in $C^\infty\left(\mathbb{R}^2\right)$, then $v \in C^\infty\left(\mathbb{R}^2, S^2\right)$ and

$$-\triangle v = |\nabla v|^2 v, \quad v^3(x) = 0, \quad |\nabla v(x)| \le c.$$

Hence $v(x) = (e^{i(c_0+c_1x^1+c_2x^2)}, 0)$. From $|\partial_1 v_j(x)|^2 - |\partial_2 v_j(x)|^2 \to 0$, $\partial_1 v_j(x)\cdot\partial_2 v_j(x) \to 0$, we know $|\partial_1 v(x)|^2 - |\partial_2 v(x)|^2 = 0$, $\partial_1 v(x)\cdot\partial_2 v(x) = 0$. Thus $c_1^2 - c_2^2 = 0$, $c_1c_2 = 0$, and consequently $c_1 = c_2 = 0$. Hence v is a constant map and we get Claim 6.6.

Claim 6.7 $|u^3(x)| \le c(u)e^{-\frac{|x|}{16}}$.

Proof of Claim 6.7 Choose $r_0 > 0$ such that $|\nabla u(x)|^2 + \left(u^3(x)\right)^2 \le \frac{3}{4}$ for $|x| \ge r_0$. For any $|x| \ge 2r_0$, we have

$$-\triangle u^3 + \left(1 - |\nabla u|^2 - \left(u^3\right)^2\right)u^3 = 0, \quad 1 - |\nabla u|^2 - \left(u^3\right)^2 \ge \frac{1}{4} \text{ on } B_{\frac{|x|}{2}}(x).$$

By comparison arguments, using the function in Lemma 4.5 we obtain $|u^3(x)| \le e^{-\frac{|x|}{16}}$. This proves Claim 6.7.

We note elliptic estimates imply that $|\nabla u^3(x)| \le c(u)e^{-\frac{|x|}{16}}$.

Claim 6.8 $|\varphi(x)| = \left|\frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}(x)\right| \le \frac{c(u)}{|x|^2}$ *for* $|x| \ge 1$.

Proof of Claim 6.8 Let $R = |x|$. By the decay property of u^3 and $|\nabla u^3|$ at ∞ and

$$\begin{aligned}\frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}(x) = &\int_{B_{\frac{R}{2}}} \frac{1}{\pi y}\frac{\partial\left(u^3\right)^2}{\partial z}(x-y)\,dy + \int_{\partial B_{\frac{R}{2}}} \frac{1}{\pi y}\frac{\overline{y}}{R}\left(u^3\right)^2(x-y)\,ds \\ &- \int_{\mathbb{R}^2\setminus B_{\frac{R}{2}}} \frac{\left(u^3\right)^2(x-y)}{\pi\left(y^1+iy^2\right)^2}dy,\end{aligned}$$

one gets

$$\left|\frac{1}{\pi z} * \frac{\partial\left(u^3\right)^2}{\partial z}(x)\right| \le c(u)Re^{-\frac{R}{32}} + \frac{c(u)}{R^2} \le \frac{c(u)}{R^2}.$$

This proves Claim 6.8.

Claim 6.9 There exists a $c(u) > 0$ such that $|\nabla u(x)| \le \frac{c(u)}{|x|}$ for $|x| \ge 1$.

Proof of Claim 6.9 Choose $x_j \to \infty$, let $r_j = |x_j|$. Define $v_j(x) = u\left(x_j + \frac{r_j}{2}x\right)$ for $x \in \overline{B_1}$. Then

$$-\triangle v_j = \left(|\nabla v_j|^2 + \frac{r_j^2}{4}\left(v_j^3\right)^2\right)v_j - \frac{r_j^2}{4}v_j^3 e_3, \quad |v_j^3(x)| \le c(u)e^{-\frac{r_j}{32}} \text{ on } B_1.$$

Let $K_j = \sup_{|x|\le 1}(1-|x|)\,|\nabla v_j(x)|$. We claim that K_j remains bounded. Otherwise, one would have a sequence of $K_j \to \infty$, and a sequence of points $y_j \in B_1$ such that $K_j = (1-|y_j|)\,|\nabla v_j(y_j)|$. Denote $\sigma_j = 1 - |y_j|$, and define $w_j(x) = v_j(y_j + \frac{\sigma_j}{K_j}x)$ for $x \in B_{K_j}$. Then

$$-\triangle w_j = \left(|\nabla w_j|^2 + \frac{\sigma_j^2 r_j^2}{4K_j^2}\left(w_j^3\right)^2\right)w_j - \frac{\sigma_j^2 r_j^2}{4K_j^2}w_j^3 e_3,$$

$$|\nabla w_j(x)| \le \frac{1}{1-\frac{|x|}{K_j}}, \quad |\nabla w_j(0)| = 1, \quad |w_j^3(x)| \le c(u)e^{-\frac{r_j}{32}}.$$

Thus $|w_j|_{C^{1,\alpha}\left(\overline{B_r}\right)} \le c(\alpha, r)$ for $0 < \alpha < 1$, $r > 0$. We may assume $w_j \to w$ in $C^\infty\left(\mathbb{R}^2\right)$; then $w \in C^\infty\left(\mathbb{R}^2, S^2\right)$ and

$$-\triangle w = |\nabla w|^2 w, \quad w^3 = 0, \quad |\nabla w(x)| \le 1, \quad |\nabla w(0)| = 1.$$

Therefore $w(x) = (e^{i(c_0 + c_1 x^1 + c_2 x^2)}, 0)$, where c_0, c_1, c_2 are real constants, $c_1^2 + c_2^2 = 1$. On the other hand,

$$\begin{aligned}||\partial_1 w_j(x)|^2 - |\partial_2 w_j(x)|^2| &= \frac{\sigma_j^2 r_j^2}{4K_j^2}\left|\left(|\partial_1 u_j|^2 - |\partial_2 u_j|^2\right)\left(x_j + \frac{r_j}{2}\left(y_j + \frac{\sigma_j}{K_j}x\right)\right)\right| \\ &\le c(u)\frac{\sigma_j^2}{K_j^2} \to 0,\end{aligned}$$

$$|\partial_1 w_j(x)\cdot\partial_2 w_j(x)| = \frac{\sigma_j^2 r_j^2}{4K_j^2}\left|\left(\partial_1 u_j \cdot \partial_2 u_j\right)\left(x_j + \frac{r_j}{2}\left(y_j + \frac{\sigma_j}{K_j}x\right)\right)\right| \le c(u)\frac{\sigma_j^2}{K_j^2} \to 0.$$

Hence $|\partial_1 w(x)|^2 = |\partial_2 w(x)|^2$, $\partial_1 w(x)\cdot\partial_2 w(x) = 0$. The latter implies $c_1^2 - c_2^2 = 0$, $c_1 c_2 = 0$, which contradicts $c_1^2 + c_2^2 = 1$, and Claim 6.9 is proved by going back from v_j to u.

The quantization of $\int_{\mathbb{R}^2}\left(u^3\right)^2 dx$ follows from Proposition 6.1.

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