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On the Extension of Isometries between Unit Spheres of E and $C(\Omega)$

Guang Gui DING
Department of Mathematics, Nankai University, Tianjin 300071*, P. R. China*
E-mail: ding_gg@nankai.edu.cn

Abstract In this paper, we study the extension of isometries between the unit spheres of some Banach spaces E and the spaces $C(\Omega)$. We obtain that if the set sm.$S_1(E)$ of all smooth points of the unit sphere $S_1(E)$ is dense in $S_1(E)$, then under some condition, every surjective isometry V_0 from $S_1(E)$ onto $S_1(C(\Omega))$ can be extended to be a real linearly isometric map V of E onto $C(\Omega)$. From this result we also obtain some corollaries. This is the first time we study this problem on different typical spaces, and the method of proof is also very different too.

Keywords Extension of isometry, Smooth point, WCG, w-Asplund space
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In [1], Tingley proposed the following problem:

"Let E and F be real normed spaces with unit spheres $S_1(E)$ and $S_1(F)$. Suppose that $T : S_1(E) \to S_1(F)$ is an onto-isometry. Is T necessarily the restriction to $S_1(E)$ of a linear, or affine, transformation on E ?" (In the complex spaces, it is evident that the answer is negative, for example, when we take $E = F = \mathbf{C}$(complex plane) and $T(x) = \overline{x}$).

And in his paper, he only got an affirmative answer to the assertion $T(-x) = -T(x)$ ($\forall x \in S_1(E)$) when the spaces are finitely dimensional. In [2], the above conclusion was also obtained when the spaces are any "strictly convex" Banach spaces.

In [3] and [4], Tingley's problem was affirmatively answered (i.e. the surjective isometry $V_0 : S_1(E) \to S_1(F)$ can be extended to be a linear map V on E) on the spaces $C_0(\Omega)$ (where Ω is a locally compact Hausdorff space which has at least two points, and for each $x \in C_0(\Omega)$ the set $\{t||x(t)| \geq \alpha, t \in \Omega\}$ is compact as $\alpha > 0$) and on the aton-generating spaces AL_p (i.e. abstract L_p spaces) ($1 < p < \infty$, $p \neq 2$).

In [5–9], some affirmative results were obtained, especially for the isometries on the unit spheres of the l^1-sum of $C_0(\Omega, E)$ and l^p-sum of $C_0(\Omega, E)$ (where, E satisfies some conditions). And in [10], the affirmative result was also obtained for the unit spheres of some subspaces of $L^1(\Omega, X)$ (where, X is a strictly convex space).

In this paper, we shall consider the above problem on some surjective isometry from $S_1(E)$ onto $S_1(C(\Omega))$ for some normed spaces E whose "smooth points" of the unit sphere sm.$[S_1(E)]$

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are dense in $S_1(E)$. This is the first time we study this problem on different typical spaces, and the method of proof is also very different too. And we can consider the problem on real or complex spaces.

First we introduce a lemma which is the generalization of Lemma 9.4.6 in [11]. (Notice that there is some mistake there, the conclusion of that lemma is valid only for real spaces. The counter-example can be taken for $X = Y = \mathbf{C}$(complex plane) and $U(x) = \overline{x}$).

Lemma *Let V_0 be an isometric map from the unit sphere $S_1(E)$ of E into the unit sphere $S_1(F)$ of F (where E and F are real normed spaces), $x_0 \in sm.[S_1(E)]$ and $g \in F^*$ be some continuous linear functional of norm one such that $g[V_0(\pm x_0)] = \pm \|x_0\| (= \pm 1)$. If for every $x \in S_1(E)$ we have*

$$\|V_0(x) - |\lambda| V_0(\pm x_0)\| \leq \|x - |\lambda|(\pm x_0)\|, \quad \forall \lambda \in R, \tag{$*$}$$

then $g(V_0(x)) = f_{x_0}(x)$, $x \in S_1(E)$ (where $f_{x_0}$is the support functional at x_0 in the unit ball of E).

Proof Suppose that for $x_1 \in S_1(E)$, $g[V_0(x_1)] \neq f_{x_0}(x_1)$. Let $\alpha = g[V_0(x_1)]$. Then we have $f_{x_0}(x_1 - \alpha x_0) \neq 0$. By Lemma 9.4.5 in [11], there exists a real γ such that

$$\|x_0 + \gamma(x_1 - \alpha x_0)\| < \|x_0\| \quad \text{(it is evident that } \gamma \neq 0).$$

Let us put $\beta = 1/\gamma$; then the above implies

$$\|x_1 - (\alpha - \beta)x_0\| < \|\beta x_0\| = |\beta| .$$

By the hypothesis of the lemma, and from the above inequality we can get:

If $\alpha - \beta = 0$, then we obtain

$$|\beta| = |\alpha| = |g(V_0(x_1))| \leq \|V_0(x_1)\| = \|x_1\| = \|x_1 - (\alpha - \beta)x_0\| < \|\beta x_0\| = |\beta| ,$$

which leads to a contradiction.

And if $\alpha - \beta \neq 0$ then we obtain

$$\begin{aligned} |\beta| = |\alpha - (\alpha - \beta)| &= \left| g(V_0(x_1)) - |\alpha - \beta| \cdot g\left[V_0 \left(\frac{\alpha - \beta}{|\alpha - \beta|} x_0 \right) \right] \right| \\ &= \left| g\left[V_0(x_1) - |\alpha - \beta| \cdot V_0 \left(\frac{\alpha - \beta}{|\alpha - \beta|} x_0 \right) \right] \right| \leq \left\| V_0(x_1) - |\alpha - \beta| \cdot V_0 \left(\frac{\alpha - \beta}{|\alpha - \beta|} x_0 \right) \right\| \\ &\leq \left\| x_1 - |\alpha - \beta| \left(\frac{\alpha - \beta}{|\alpha - \beta|} x_0 \right) \right\| = \|x_1 - (\alpha - \beta)x_0\| < \|\beta x_0\| = |\beta| , \end{aligned}$$

which leads to a contradiction too. Thus we complete the proof.

The main theorem is as follows:

Theorem 1 *Let E be a normed space, the subset $sm.[S_1(E)]$ of the smooth points of the unit sphere of E be dense in its unit sphere $S_1(E)$, V_0 be an isometric map from $S_1(E)$ onto the unit* *sphere $S_1(C(\Omega))$ (where, Ω is a compact Hausdorff space). If for every $x \in S_1(E)$ and $x_0 \in sm.[S_1(E)]$, we have*

$$\|V_0(x) - |\lambda|\, V_0(x_0)\| \le \|x - |\lambda|\, x_0\|\,, \quad \forall \lambda \in \mathbf{R}.$$

Then V_0 must uniquely have a real linearly isometric extension V on the whole E. Moreover, if we also have $V_0(ix) = iV_0(x)$ (in the case of complex spaces), then the above V is linear.

Proof Let

$$V(x) = \begin{cases} \|x\|\, V_0\left(\dfrac{x}{\|x\|}\right), & \text{if } x \ne \theta, \\ \theta, & \text{if } x = \theta. \end{cases}$$

Then, by the hypothesis that V_0 is isometric on $S_1(E)$, we have

$$\|V(x_1) - V(x_2)\| = \|x_1 - x_2\|\,, \text{ if } \|x_1\| = \|x_2\| \quad (\forall x_1, x_2 \in E). \tag{1}$$

Since $V_0(S_1(E)) = S_1(C(\Omega))$, we have

$$\|V(x)\| = \|x\| \quad (\forall x \in E) \tag{2}$$

and

$$\|V(\alpha_0 x) - V(\alpha x)\| = (\alpha_0 - \alpha)\,\|x\| \quad (\forall 0 < \alpha < \alpha_0,\ x \in E). \tag{3}$$

We shall now prove the following two conclusions:

(i) For each $x \in E$, if there exist $t_0 \in \Omega$ and $\theta_0 \in \mathbf{C}$ with $|\theta_0| = 1$ such that $\theta_0 V(\alpha_0 x)(t_0) = \alpha_0\,\|x\|$, then we assert that $\theta_0 V(\alpha x)(t_0) = \alpha\,\|x\|$ for each positive α with $\alpha < \alpha_0$.

(ii) For each $x \in E$, if there exist $t_0 \in \Omega$ and $\theta_0 \in \mathbf{C}$ with $|\theta_0| = 1$ such that $\theta_0[V(x) - V(-x)](t_0) = \|V(x) - V(-x)\|$, then we assert that $\theta_0 V(\pm x)(t_0) = \pm\,\|x\|$.

In fact, to check (i) is easy. From (2) and (3) above, we have

$$\begin{aligned} \theta_0[V(\alpha_0 x)(t_0) - V(\alpha x)(t_0)] &= \alpha_0\,\|x\| - \theta_0 V(\alpha x)(t_0) \\ &\ge \alpha_0\,\|x\| - \|V(\alpha x)\| = \alpha_0\,\|x\| - \|\alpha x\| = (\alpha_0 - \alpha)\,\|x\| \end{aligned}$$

and

$$\theta_0[V(\alpha_0 x)(t_0) - V(\alpha x)(t_0)] \le \|V(\alpha_0 x) - V(\alpha x)\| = (\alpha_0 - \alpha)\,\|x\|\,.$$

So we get that $\theta_0 V(\alpha x)(t_0) = \alpha\,\|x\|$.

To check (ii), by (1), we have

$$\|V(x) - V(-x)\| = \|x - (-x)\| = 2\,\|x\|\,,$$

and by (2), we have

$$|\theta_0 V(\pm x)(t_0)| \le \|V(\pm x)\| = \|x\|\,.$$

Noticing the hypothesis in (ii), the two relations above imply

$$|\theta_0 V(\pm x)(t_0)| = \|x\|\,.$$

If $\|x\| \neq 0$, and there exists $\theta_1 \in \mathbf{C}$ with $|\theta_1| = 1$ such that $\theta_1\theta_0 V(x)(t_0) = \|x\|$, then we have, by the hypothesis of (ii), that

$$\frac{\|x\|}{\theta_1} - \theta_0 V(-x)(t_0) = \|V(x) - V(-x)\| = 2\,\|x\|\,.$$

Hence we have

$$\left|2 - \frac{1}{\theta_1}\right| \|x\| \leq \|V(-x)\| = \|-x\| = \|x\|\,,$$

and we get

$$\left|2 - \frac{1}{\theta_1}\right| \leq 1.$$

Notice that for each $\theta \in \mathbf{C}$ with $|\theta| = 1$ and $\theta \neq 1$ it must be true that $|2 - \theta| > 1$. Thus we obtain from the above inequality that $\theta_1 = 1$, i.e. $\theta_0 V(x)(t_0) = \|x\|$. We can also get that $\theta_0 V(-x)(t_0) = -\|x\|$ by the same method of proof.

By (i) and (ii) above, we can get that for each $x \in E$ there exist a sequence $\{\mathrm{t}_n\} \subset \Omega$ and a sequence $\{\theta_n\} \subset \mathbf{C}$ with $|\theta_n| = 1$ such that for every $\lambda \in \mathbf{R}$, if $|\lambda| \leq n_1 (\in \mathbf{N})$ then we have

$$\theta_n V(\lambda x)(t_n) = \lambda\,\|x\|\,, \quad \forall n \geq n_1. \tag{4}$$

From the compactness of Ω and the unit sphere of $\mathbf{C}$, the sequences $\{t_n\}$ and $\{\theta_n\}$ have the cluster points $t_x \in \Omega$ and θ_x with $|\theta_x| = 1$, respectively. Hence, by the continuity of $V(\lambda x)$, we have $\theta_x V(\lambda x)(t_x) = \lambda\,\|x\|$ (notice that the t_x and θ_x are both independent of λ from the above relations (i) and (ii)).

That is, we obtain

$$\theta_x \delta_{t_x}[V(\lambda x)] = \lambda\,\|x\|\,, \quad \forall \lambda \in \mathbf{R},\ (\forall x \in E) \tag{5}$$

(where $\delta_{t_x} \in C(\Omega)^*$ and $\delta_{t_x}(y) = y(t_x)$, $\forall y \in C(\Omega)$).

In the case of complex spaces, we can consider them as the linear spaces over reals (notice that the set of all smooth points sm.$S_1(E)$ remains unchanged). Since $\theta_x\delta_{t_x} \in C\,(\Omega)^*$, when we see $C\,(\Omega)$ as a real space, the rel.$(\theta_x\delta_{t_x})$ (the real part of the functional $\theta_x\delta_{t_x}$) is a continuous real linear functional of norm one on $C\,(\Omega)$. Notice that if $x_0 \in$ sm.$S_1(E)$, then the corresponding $g_0 =$ rel.$(\theta_{x_0}\delta_{t_{x_0}})$ also satisfies the above (5). By the above lemma, we can obtain that for each smooth point x_0 in the unit sphere $S_1(E)$, there exists a unique real $f_{x_0} \in E^*$, which is the "support functional" at x_0 (i.e., $\|f_{x_0}\| = 1$ and $f_{x_0}(x_0) = \|x_0\|$) such that rel.$(\theta_x\delta_{t_x}) \circ V = f_{x_0}$. That is, for each smooth point (element) $x_0 \in S_1(E)$, there exists a point $t_{x_0} \in \Omega$ such that $[V(x)](t_{x_0})$ is real linear for all $x \in E$.

Since $[V(x)](t)$ is continuous on Ω, in order to prove that V is real linear it is enough that the above subset $\{t_{x_0} | x_0 \in \mathrm{sm}.S_1(E)\}$ is dense in Ω. We shall now prove this as follows.

For every $t_1 \in \Omega$ and each open neighborhood U_{t_1} of t_1, there exists an open neighborhood $V_{t_1}$of t_1 such that $\overline{V_{t_1}} \subset U_{t_1}$, since the compact Ω is a regular topological space. By Uryson's lemma (noticing Ω is also a normal topological space), we can get a continuous function $y_1(t)$ on Ω such that

$$y_1(t)|_{\overline{V_{t_1}}} \equiv 1, \quad y_1(t)|_{U^c_{t_1}} \equiv 0, \ \text{and} \ |y(t)| \leq 1, \quad \forall t \in \Omega.$$

By the hypothesis that V_0 is a surjective isometry from $S_1(E)$ onto $S_1(C(\Omega))$, there exists a unique element $x_1 \in S_1(E)$ such that $V(x_1) = V_0(x_1) = y_1$. Since the hypothesis that $\overline{\mathrm{sm}.S_1(E)} = S_1(E)$ and V is a continuous and there is a "one to one" map from $S_1(E)$ onto $S_1[C(\Omega)]$, we have

$$\overline{V[\mathrm{sm}.S_1(E)]} = V[\overline{\mathrm{sm}.S_1(E)}] = V[S_1(E)] = S_1[C(\Omega)].$$

Hence, for $y_1 \in S_1[C(\Omega)]$ and some $\varepsilon \in (0,1)$, there exists $x_0 \in \mathrm{sm}.S_1(E)$, such that

$$\|V(x_0) - y_1\| < \varepsilon.$$

Thus we get

$$|V(x_0)(t)| < |y_1(t)| + \varepsilon = 0 + \varepsilon = \varepsilon, \quad \forall t \in U_{t_1}^c.$$

Noticing that $\|V(x_0)\| = \|x_0\| = 1$, the above inequality implies that the corresponding point t_{x_0} which satisfies the above (5) must be in the neighborhood U_{t_1} of t_1. Hence the above subset $\{t_{x_0} | x_0 \in \mathrm{sm}.S_1(E)\}$ is dense in Ω and $V(x)$ is real linear on E. Thus V is an isometry of E onto $C(\Omega)$ by the above (2), and is linear by the hypothesis of $V_0(ix) = iV_0(x)$ (in the case of complex spaces).

Remark 1 In the above proof, the relation (5) can be obtained easily by using the positive-homogeneity of $V(x)$. However, the advantage of the above proof is that it is also valid for any non-linear isometry.

Remark 2 Let V be a non-linear isometry of a normed space E into $C(\Omega)$ (where Ω is a compact Hausdorff space). If we take $g = \mathrm{rel}.(\theta_{x_0}\delta_{t_{x_0}})$ in the proof of Theorem 1, the above lemma is still valid without the condition $(*)$ by the relation (5). Therefore, we can get the following proposition from the proof of Theorem 1.

Proposition *Let V be a (non-linear) isometry of a normed space E into $C(\Omega)$ with $V(\theta) = \theta$. Then, for each smooth point x_0 of the unit sphere $S_1(E)$, there exists a point $t_{x_0} \in \Omega$ such that $[V(x)](t_{x_0})$ is the real linear functional f_{x_0} (where f_{x_0} is the "support functional" at x_0).*

Now we give some corollaries, as follows:

Corollary 1 *If $E = L^1(\Omega, \mu)$ (where (Ω, μ) is "σ-finite") in Theorem 1 above, then the conclusion of Theorem 1 is valid.*

Proof In this case, we have that $(L^1(\Omega, \mu))^* \cong L^\infty(\Omega, \mu)$, and we can get (see, for example, p. 170 in [12])

$$x_0 \in \mathrm{sm}.[S_1(L^1(\Omega, \mu))] \Leftrightarrow \mu(\{t | x_0(t) = 0\}) = 0, \quad \forall x_0 \in L^1(\Omega, \mu).$$

Hence it is easy to verify that $\overline{\mathrm{sm}.[S_1(L^1(\Omega, \mu))]} = S_1(L^1(\Omega, \mu))$.

Corollary 2 *If we put $E = C(\Omega_1)$ (where Ω_1 is compact Hausdorff space) or $E = l^\infty$ in Theorem 1, then the conclusion of Theorem 1 is still valid. In particular (for real spaces), the isometry V_0 from the unit sphere $S_1(C(\Omega))$ onto itself must have a linearly isometric extension V on the whole space $C(\Omega)$, and so is the space l^∞.*

Proof In $C(\Omega_1)$ we have that $x_0 \in \text{sm.}[S_1(\Omega_1)] \Leftrightarrow x_0$ is a "peak function" on Ω_1 ($\forall x_0 \in X(\Omega_1)$), (where x_0 is called a peak function if there exists only one point $t_0 \in \Omega_1$ such that $|x_0(t_0)| = \|x_0\|$) [12]. Hence it is easy to verify that $\overline{\text{sm.}[S_1(\Omega_1)]} = S_1(\Omega_1)$.

And if $E = l^\infty$, we notice only that $l^\infty \cong C(\beta(\mathbf{N}))$ (where, $\beta(\mathbf{N})$ is the Stone–Cech compactification of $\mathbf{N}$, see, for example, p. 113 in [12]), then we can get the conclusion from the above statement.

Corollary 3 *If E is a separable or reflexive Banach space, then the conclusion of the above Theorem* 1 *is valid.*

Proof By the Mazur density theorem (see, for example, p. 171 in [12]), we have that if E is a separable Banach space then $\text{sm.}[S_1(E)]$ is a residual subset of $S_1(E)$, and the above conclusion is also valid if E is a reflexive Banach space (see, p. 172 in [12]). Hence this completes the proof.

Corollary 4 *If E is a Larman–Phelps space* (*i.e. the GDS*), *in particular, if E is a weak-Asplund space, then the conclusion of the above Theorem* 1 *is valid.*

Proof We notice only that the Larman–Phelps space is a Banach space which satisfies that every continuous convex function defined on an open convex subset of E is Gateaux-differentiable in a dense subset of its domain, the w-Asplund space is the above space in which the above function must be Gateaux-differentiable in a dense G_δ-set in its domain [13]. Moreover, $\|x\|$ is Gateaux-differentiable at $x_0 \in S_1(E) \Leftrightarrow x_0$ is a smooth point in $S_1(E)$ [13, 14].

Corollary 5 *If E is a WCG* (*weakly compactly generated*) *space, in particular, if E is the space $c_0(\Gamma)$* (*where Γ is any index set*), *then the conclusion of the above Theorem* 1 *is valid.*

Proof It is enough to notice that each WCG space is a w-Asplund space [13] and the above $c_0(\Gamma)$ is a WCG space (see, for example, p. 143 in [14]) (in fact, the $c_0(\Gamma)$ is an Asplund space since $(c_0(\Gamma))^* \cong l_1(\Gamma)$ has Radon–Nikodym property [13]).

At the end of this paper, by the same method of proof in Theorem 1 above, we can get the following theorem:

Theorem 2 *In Theorem* 1, *if we replace $C(\Omega)$ with c_0, then the conclusion of Theorem* 1 *above is valid.*

Proof We notice only that, in the space c_0, if $\theta \neq x \in c_0$ and $x = \{x(k)\}$ then there is $k_x \in \mathbf{N}$ such that $|x(k_x)| = \|x\|$. Hence, by the same method of proof as in Theorem 1, there exist a sequence $\{k_n\} \subset \mathbf{N}$ and a sequence $\{\theta_n\} \subset \mathbf{C}$ with $|\theta_n| = 1$ such that for each $\lambda \in R$, if $|\lambda| \le n_1$ ($\in \mathbf{N}$) then we have

$$\theta_n V(\lambda x)(k_n) = \lambda \|x\|, \quad \forall n \ge n_1.$$

Since $V(\lambda x)(k) \to 0$ as $k \to 0$, we can get a $k_x \in \mathbf{N}$ and $\theta_x \in \mathbf{C}$ with $|\theta_x| = 1$ such that

$$\theta_x V(\lambda x)(k_x) = \lambda \|x\| \quad (\forall \lambda \in \mathbf{R},\ \forall \mathbf{x} \in \mathbf{E}).$$

Using the proof of Theorem 1 as well as the above Lemma, we know that for each smooth point (element) $x_0 \in S_1(E)$, there exists a natural number $k_{x_0} \in \mathbf{N}$ such that the coordinate $[V(x)](k_{x_0})$ is real linear for all $x \in E$.

In order to prove that V is real linear it is enough that the above subset $\{k_{x_0} | x_0 \in \text{sm}.S_1(E)\} = \mathbf{N}$. We shall now prove this as follows:

For each $k_1 \in \mathbf{N}$, we take the element $e_{k_1} = (0, \ldots, 0, 1, 0, \ldots)$ (where the k_1-th coordinate is 1). For the element e_{k_1} and some $\varepsilon \in (0,1)$, by the same deduction which was used in Theorem 1 above, there exists $x_0 \in \text{sm}.S_1(E)$, such that

$$\|V(x_0) - e_{k_1}\| < \varepsilon,$$

which implies

$$|V(x_0)(k)| < |e_{k_1}(k)| + \varepsilon = 0 + \varepsilon = \varepsilon < 1, \quad \forall k \neq k_1.$$

Noticing that $\|V(x_0)\| = \|x_0\| = 1$, the above inequality implies that $k_1 = k_{x_0}$. Thus we complete the proof.

Finally, we can immediately get the following corollary from Theorem 2 above:

Corollary 6 *Let V_0 be an isometric map from the unit sphere $S_1(c_0)$ of c_0 onto itself. If for every $x \in S_1(c_0)$, and $x_0 \in \text{sm}.[S_1(c_0)]$, we have*

$$\|V_0(x) - |\lambda|\, V_0(x_0)\| \leq \|x - |\lambda|\,(x_0)\|\,, \quad \forall \lambda \in \mathbf{R},$$

then V_0 must have a real linear isometric extension V on the whole space c_0. Moreover, if we also have $V_0(ix) = iV_0(x)$ (in the case of complex spaces), then V above is linear.

Remark There are some examples to show that a mapping V_0 from the unit sphere $S_1(E)$ to the unit sphere $S_1(E_1)$ does not need to be an isometry even if V_0 is continuous and bijective. For example (like an example which was made by Wang Risheng), in the real space $C(a,b)$, we define the map V_0 on the unit sphere $S_1(C(a,b))$ as follows:

$$V_0(x) = x(t) + \frac{1}{3\pi}\sin[2\pi \cdot x(t)], \quad \forall x = x(t) \in S_1(C(a,b)).$$

Notice that the real function $v(\xi) = \xi + \frac{1}{3\pi}\sin(2\pi\xi)$ satisfies that

$$v'(\xi) = 1 + \frac{2\pi}{3\pi}\cos(2\pi\xi) > 0, \quad \forall \xi \in \mathbf{R}.$$

Hence, $v(\xi)$ is a "one to one" and strictly increasing function, and we obtain that V_0 is a bijective map from $S_1(C(a,b))$ onto itself since $v(\pm 1) = \pm 1$. But V_0 is not isometric and the following mapping V :

$$V(x) = \|x\| \left\| T\left(\frac{x}{\|x\|}\right) \right\| = x(t) + \frac{\|x\|}{3\pi}\sin\left[2\pi \cdot \frac{x(t)}{\|x\|}\right], \quad \forall x = x(t) \in C(a,b)$$

(where, we define that $V(x) = \theta$, if $x = \theta$), is not a linearly isometric map of x on the space $C(a,b)$.

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