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Semi-Fredholm Spectrum and Weyl's Theorem for Operator Matrices

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Abstract When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C an operator acting on the Hilbert space $H \oplus K$ of the form $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$. In this paper, first we give the necessary and sufficient condition for M_C to be an upper semi-Fredholm (lower semi-Fredholm, or Fredholm) operator for some $C \in B(K, H)$. In addition, let $\sigma_{SF_+}(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is not an upper semi-Fredholm operator$\}$ be the upper semi-Fredholm spectrum of $A \in B(H)$ and let $\sigma_{SF_-}(A) = \{\lambda \in \mathbb{C} : A - \lambda I$ is not a lower semi-Fredholm operator$\}$ be the lower semi-Fredholm spectrum of A. We show that the passage from $\sigma_{SF_\pm}(A) \cup \sigma_{SF_\pm}(B)$ to $\sigma_{SF_\pm}(M_C)$ is accomplished by removing certain open subsets of $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$ from the former, that is, there is an equality

$$\sigma_{SF_\pm}(A) \cup \sigma_{SF_\pm}(B) = \sigma_{SF_\pm}(M_C) \cup \mathscr{G},$$

where $\mathscr{G}$ is the union of certain of the holes in $\sigma_{SF_\pm}(M_C)$ which happen to be subsets of $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$. Weyl's theorem and Browder's theorem are liable to fail for 2×2 operator matrices. In this paper, we also explore how Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem survive for 2×2 upper triangular operator matrices on the Hilbert space.

Keywords Semi-Fredholm operator, Fredholm operator, Spectrum, Weyl's theorem
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1 Introduction

The study of upper triangular operator matrices arises naturally from the following fact: If A is a Hilbert space operator and M is an invariant subspace for A, then A has the following 2×2 upper triangular operator matrix representation:

$$A = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} : \quad M \oplus M^\perp \longrightarrow M \oplus M^\perp,$$

and one way to study operators is to see them as entries of simpler operators. The upper triangular operator matrices have been studied by many authors (such as [1–5], etc.). This paper is concerned with the semi-Fredholm spectrum and essential spectrum of 2×2 upper triangular operator matrices. We also study Weyl's theorem and a-Weyl's theorem for 2×2 upper triangular operator matrices.

Throughout this paper, let H and K be infinite-dimensional separable Hilbert spaces, let $B(H, K)$ denote the set of bounded linear operators from H to K, with $B(H, H)$ abbreviated to $B(H)$. If $A \in B(H)$, write $N(A)$ for the null space of A and $R(A)$ for the range of A.

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For $A \in B(H)$, if $R(A)$ is closed and $\dim N(A) < \infty$, we call A an upper semi-Fredholm operator and if $\dim H/R(A) < \infty$, then A is called a lower semi-Fredholm operator. Let $\Phi_+(H)$ ($\Phi_-(H)$) denote the set of all upper (lower) semi-Fredholm operators on H. A is called a Fredholm operator if $\dim N(A) < \infty$ and $\dim H/R(A) < \infty$. If A is a semi-Fredholm operator, letting $n(A) = \dim N(A)$ and $d(A) = \dim H/R(A)$, then we define the index of A by $\operatorname{ind}(A) = n(A) - d(A)$. An operator A is called Weyl if it is a Fredholm operator of index zero, and is called Browder if it is Fredholm "of finite ascent and descent". We write $\alpha(A)$ for the ascent of $A \in B(H)$. Let A^* denote the conjugate of $A \in B(H)$. If $A \in B(H)$, write $\sigma(A)$ for the spectrum of A; $\sigma_a(A)$ for the approximate point spectrum of A; $\pi_{00}(A)$ for the isolated points of $\sigma(A)$ which are eigenvalues of finite multiplicity; $\pi_{00}^a(A)$ for the isolated points of $\sigma_a(A)$ which are eigenvalues of finite multiplicity. Let $\rho_a(A) = \mathbb{C} \backslash \sigma_a(A)$. The essential spectrum $\sigma_e(A)$, the Weyl spectrum $\sigma_w(A)$, the Browder spectrum $\sigma_b(A)$ of A are defined by: $\sigma_e(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Fredholm}\}$; $\sigma_w(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Weyl}\}$; $\sigma_b(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not Browder}\}$.

For any $A \in B(H)$, let

$$\sigma_{SF_+}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not in } \Phi_+(H)\},$$
$$\sigma_{SF_-}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \text{ is not in } \Phi_-(H)\}.$$

We call $\sigma_{SF_+}(A)$ and $\sigma_{SF_-}(A)$ upper semi-Fredholm spectrum and lower semi-Fredholm spectrum of A, respectively.

Recall that an operator $A \in B(H)$ is said to be bounded below if there is a $k > 0$ for which $\|x\| \leq k\ \|Ax\|$ for each $x \in H$. A is bounded below if and only if $0 \in \rho_a(A)$. If $\mathscr{G}$ is a compact subset of $\mathbb{C}$, we write int $\mathscr{G}$ for the interior points of $\mathscr{G}$; iso $\mathscr{G}$ for the isolated points of $\mathscr{G}$; $\partial\, \mathscr{G}$ for the topological boundary of $\mathscr{G}$. When $A \in B(H)$ and $B \in B(K)$ are given, we denote by M_C an operator acting on $H \oplus K$ of the form

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix},$$

where $C \in B(K, H)$.

In [1] and [2], the authors gave the necessary and sufficient condition for M_C to be invertible for some $C \in B(K, H)$ and characterized the spectrum of M_C. In Section 2 in this paper, we give the necessary and sufficient condition for M_C to be an upper semi-Fredholm operator (lower semi-Fredholm or Fredholm) operator for some $C \in B(K, H)$ and characterize the semi-Fredholm spectrum and essential spectrum of M_C.

In Section 3, we show the passage from $\sigma_{SF_\pm}(A) \cup \sigma_{SF_\pm}(B)$ $(\sigma_e(A) \cup \sigma_e(B))$ to $\sigma_{SF_\pm}(M_C)$ $(\sigma_e(M_C))$ can be described as follows:

$$\sigma_{SF_\pm}(A) \cup \sigma_{SF_\pm}(B) = \sigma_{SF_\pm}(M_C) \cup \mathscr{G}, \quad \sigma_e(A) \cup \sigma_e(B) = \sigma_e(M_C) \cup \mathscr{G},$$

where $\mathscr{G}$ lies in certain holes in $\sigma_{SF_\pm}(M_C)$ $(\sigma_e(M_C))$, which happen to be subsets of $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$.

In Section 4, we explore how Weyl's theorem, Browder's theorem, a-Weyl's theorem and a-Browder's theorem survive for 2×2 upper triangular operator matrices M_C. Weyl's theorem for operator matrices was studied in [5]. We have an example to show that our result is not compatible with the main theorem in [5].

2 Semi-Fredholm Spectrum for Operator Matrices

Lemma 2.1 *An operator $A \in B(H)$ is upper semi-Fredholm if and only if A^*A is Fredholm.*

Proof It is obvious.

In this section, our main results are:

Theorem 2.2 *A 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is an upper semi-Fredholm operator for* *some $C \in B(K,H)$ if and only if A is an upper semi-Fredholm operator and*:

$$\begin{cases} n(B) < \infty \text{ or } n(B) = d(A) = \infty, & \text{if } R(B) \text{ is closed;} \\ d(A) = \infty. & \text{if } R(B) \text{ is not closed.} \end{cases}$$

Proof We first claim that if $A \in \Phi_+(H)$ and $R(B)$ is closed, then

$$n(B) < \infty \text{ or } n(B) = d(A) = \infty \Longleftrightarrow M_C \in \Phi_+(H \oplus K) \text{ for some } C \in B(K,H).$$

If $R(A)$ and $R(B)$ are closed, then M_C as an operator from $H \oplus K = (N(A) \oplus R(A^*)) \oplus (N(B) \oplus R(B^*)) = N(A) \oplus R(A^*) \oplus N(B) \oplus R(B^*)$ into $H \oplus K = (N(A^*) \oplus R(A)) \oplus (N(B^*) \oplus R(B)) = N(A^*) \oplus R(A) \oplus N(B^*) \oplus R(B)$ has the following operator matrix:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_{11} & C_{12} \\ 0 & A_1 & C_{21} & C_{22} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix}.$$

Clearly, A_1 and B_1 are invertible. So M_C is upper semi-Fredholm if and only if the operator

$$M'_C = \begin{pmatrix} 0 & 0 & C_{11} & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix}$$

is upper semi-Fredholm. By Lemma 2.1, M'_C is upper semi-Fredholm if and only if $M'^*_C M'_C$ is Fredholm, but

$$M'^*_C M'_C = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1^* & 0 & 0 \\ C_{11}^* & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1^* \end{pmatrix} \begin{pmatrix} 0 & 0 & C_{11} & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & B_1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1^*A_1 & 0 & 0 \\ 0 & 0 & C_{11}^*C_{11} & 0 \\ 0 & 0 & 0 & B_1^*B_1 \end{pmatrix}$$

is an operator on $H \oplus K = N(A) \oplus R(A^*) \oplus N(B) \oplus R(B^*)$. Noticing the fact above, the first part of Theorem 2.2 is easy verified. In fact, to end the proof, what remains is to check under which conditions the operators $A_1^*A_1$, $C_{11}^*C_{11}$ and $B_1^*B_1$ are Fredholm.

If $M_C \in \Phi_+(H \oplus K)$ for some $C \in B(K,H)$ and $R(B)$ is closed, then $A \in \Phi_+(H)$ and $C_{11} \in \Phi_+(N(B), N(A^*))$. Without loss of generality, we suppose that $n(B) = \infty$, then we need to prove that $d(A) = \infty$. If not, then $n(A^*) < \infty$. It induces that $C_{11}N(B)$ is finite-dimensional. Then $N(C_{11})$ must contain an orthonormal sequence $\{y_n\}_{n=1}^{\infty}$ in $N(B)$. It is in contradiction to the fact that $n(C_{11}) < \infty$.

Conversely, suppose that $n(B) < \infty$ or $n(B) = d(A) = \infty$. If $n(B) < \infty$, then $B \in \Phi_+(K)$. By $M_C = \left(\begin{smallmatrix} I & 0 \\ 0 & B \end{smallmatrix}\right)\left(\begin{smallmatrix} I & C \\ 0 & I \end{smallmatrix}\right)\left(\begin{smallmatrix} A & 0 \\ 0 & I \end{smallmatrix}\right)$, we know that $M_C \in \Phi_+(H \oplus K)$ for every $C \in B(K,H)$. In the following, suppose $n(B) = d(A) = \infty$. Since $N(B)$ and $R(A)^\perp$ are separable, there exists a linear operator C_{11} with domain $N(B)$ and range $R(A)^\perp = N(A^*)$ such that $\|C_{11}y\| = \|y\|$ for every $y \in N(B)$. Define an operator $C : K \longrightarrow H$ by

$$C = \begin{pmatrix} C_{11} & 0 \\ 0 & 0 \end{pmatrix} : \begin{pmatrix} N(B) \\ R(B^*) \end{pmatrix} \longrightarrow \begin{pmatrix} N(A^*) \\ R(A) \end{pmatrix}. \quad (1)$$

Clearly, $C_{11}^*C_{11}$ is Fredholm and hence M_C is Fredholm. By the way, in this case, we can show that $n(M_C) = n(A) < \infty$.

Next we claim that if $A \in \Phi_+(H)$ and $R(B)$ is not closed, then

$$d(A) = \infty \Longleftrightarrow M_C \in \Phi_+(H \oplus K) \text{ for some } C \in B(K,H).$$

If $R(A)$ is closed and $R(B)$ is not closed, we first attend the following fact.

If $R(B)$ is not closed, $B = UP$ is the polar decomposition and $P = \int_0^{\|B\|} \lambda dE_\lambda$ is the spectral representation of P. $R(B)$ is not closed implies that $\dim E([0,\delta))K$ is infinite for each small enough $\delta > 0$. In this case, there exists a slight discrepancy for the space decomposition. M_C as an operator from $H \oplus K = N(A) \oplus R(A^*) \oplus E([0,\delta))K \oplus E([\delta, \|B\|])K$ into $H \oplus K = N(A^*) \oplus R(A) \oplus (N(B) \oplus UE([0,\delta))K) \oplus UE([\delta, \|B\|])K$ has the following operator matrix:

$$M_C = \begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & 0 & C_{11} & C_{12} \\ 0 & A_1 & C_{21} & C_{22} \\ 0 & 0 & B_{11} & 0 \\ 0 & 0 & 0 & B_{22} \end{pmatrix}.$$

It is easy to see that A_1 and B_{22} are invertible, hence M_C is upper semi-Fredholm if and only if the operator

$$M_C' = \begin{pmatrix} 0 & 0 & C_{11} & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & B_{11} & 0 \\ 0 & 0 & 0 & B_{22} \end{pmatrix}$$

is upper semi-Fredholm. By Lemma 2.1, M_C' is upper semi-Fredholm if and only if $M_C'^* M_C'$ is Fredholm, but

$$\begin{aligned} M_C'^* M_C' &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1^* & 0 & 0 \\ C_{11}^* & 0 & B_{11}^* & 0 \\ 0 & 0 & 0 & B_{22}^* \end{pmatrix} \begin{pmatrix} 0 & 0 & C_{11} & 0 \\ 0 & A_1 & 0 & 0 \\ 0 & 0 & B_{11} & 0 \\ 0 & 0 & 0 & B_{22} \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & A_1^* A_1 & 0 & 0 \\ 0 & 0 & C_{11}^* C_{11} + B_{11}^* B_{11} & 0 \\ 0 & 0 & 0 & B_{22}^* B_{22} \end{pmatrix} \end{aligned}$$

is an operator on $H \oplus K = N(A) \oplus R(A^*) \oplus E([0,\delta))K \oplus E([\delta, \|B\|])K$. So, to complete the proof, it is enough to verify under which conditions the operators $A_1^* A_1$, $C_{11}^* C_{11} + B_{11}^* B_{11}$ and $B_{22}^* B_{22}$ are Fredholm.

If $M_C \in \Phi_+(H \oplus K)$ for some $C \in B(K, H)$ and $R(B)$ is not closed, then $A \in \Phi_+(H)$ and $C_{11}^* C_{11} + B_{11}^* B_{11}$ is Fredholm. We need to prove that $d(A) = \infty$. If not, then $N(A^*)$ is finite-dimensional, which means that C_{11} is a finite rank operator. Therefore $C_{11}^* C_{11}$ is compact. By the perturbation theory of the Fredholm operator, we get that $B_{11}^* B_{11}$ is Fredholm. Then B_{11} is upper semi-Fredholm, it induces that B is upper semi-Fredholm, which is in contradiction to the fact that $R(B)$ is not closed. Now we have proved that $d(A) = \infty$.

Conversely, suppose that $R(B)$ is not closed and $d(A) = \infty$. By $\dim R(A)^\perp = \infty$, there exists an isometrically isomorphic linear operator $T: K \longrightarrow R(A)^\perp = N(A^*)$. Define an operator $C: K \longrightarrow H$ by

$$C = \begin{pmatrix} T \\ 0 \end{pmatrix} : K \longrightarrow \begin{pmatrix} R(A)^\perp \\ R(A) \end{pmatrix}.$$

Then $M_C \in \Phi_+(H \oplus K)$. In fact, let $\binom{u}{v} \in N(M_C)$. Then $Au + Cv = 0$ and hence $Au = -Cv \in R(A) \cap R(A)^\perp$. Thus $Au = Cv = 0$. C is injective, then $v = 0$. Therefore $N(M_C) \subseteq N(A) \oplus \{0\}$. It follows that $n(M_C) \le n(A) < \infty$ and hence $n(M_C) = n(A)$. Suppose $M_C \binom{u_n}{v_n} \longrightarrow \binom{u_0}{v_0}$. Then $Au_n + Cv_n \longrightarrow u_0$ and $Bv_n \longrightarrow v_0$. Thus $\{Au_n\}$ and $\{Cv_n\}$ are Cauchy sequences. It follows that v_n is a Cauchy sequence. Let $v_n \longrightarrow y_0$ and $Au_n \longrightarrow Ax_0$. Then $\binom{u_0}{v_0} =$

$M_C\binom{x_0}{y_0} \in R(M_C)$, which means that $R(M_C)$ is closed. Then $M_C \in \Phi_+(H \oplus K)$. The proof is completed.

A similar idea can be used to complete the proof of the following theorem:

Theorem 2.3 *A 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a lower semi-Fredholm operator for some $C \in B(K, H)$ if and only if B is a lower semi-Fredholm operator and*:

$$\begin{cases} d(A) < \infty \text{ or } d(A) = n(B) = \infty, & \text{if } R(A) \text{ is closed;} \\ n(B) = \infty, & \text{if } R(A) \text{ is not closed.} \end{cases}$$

From Theorems 2.2 and 2.3, we have

Theorem 2.4 *A 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a Fredholm operator for some $C \in B(K, H)$ if and only if A is an upper semi-Fredholm operator, B is a lower semi-Fredholm operator and one of the following cases exists*:

(a) $d(A) < \infty$ *and* $n(B) < \infty$; (b) $d(A) = n(B) = \infty$.

Proof Suppose that there exists $C \in B(K, H)$ such that M_C is Fredholm. Then $A \in \Phi_+(H)$, $B \in \Phi_-(K)$ and A is Fredholm $\Longleftrightarrow$ B is Fredholm and therefore Case (a) or Case (b) exists.

For the converse, suppose $A \in \Phi_+(H)$ and $B \in \Phi_-(K)$.

(I) Suppose that $d(A) < \infty$ and $n(B) < \infty$. Then A and B are Fredholm, hence for each $C \in B(K, H)$, M_C is Fredholm.

(II) Suppose that $d(A) = n(B) = \infty$. Define $C \in B(K, H)$ as (1) in Theorem 2.2. Then M_C is an upper semi-Fredholm operator. In order to prove that M_C is Fredholm, we need to prove that $n(M_C^*) < \infty$. Let $\binom{u}{v} \in N(M_C^*)$. Then $A^*u = 0$ and $C^*u + B^*v = 0$. Since $u \in N(A^*) = R(A)^\perp$ and $C^* = \left(\begin{smallmatrix} C_{11}^* & 0 \\ 0 & 0 \end{smallmatrix}\right)$, by the definition of C_{11}, it follows that $C^*u = C_{11}^*u \in N(B) = R(B^*)^\perp$. Then $C_{11}^*u = C^*u = -B^*v = 0$. Thus $u = 0$ and $v \in N(B^*)$. Now we get that $N(M_C^*) \subseteq \{0\} \oplus N(B^*)$, therefore $n(M_C^*) \leq n(B^*) = d(B) < \infty$. Then M_C is Fredholm.

From the proof of Theorem 2.4, we find that:

Corollary 2.5 *A 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is invertible for some $C \in B(K, H)$ if and only if A is bounded below, B is surjective, and $d(A) = n(B)$.*

Corollary 2.6 *A 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is an upper semi-Fredholm operator for some $C \in B(K, H)$ and $n(M_C) = n(A)$ if and only if A is an upper semi-Fredholm operator and*:

$$\begin{cases} n(B) \leq d(A), & \text{if } R(B) \text{ is closed;} \\ d(A) = \infty, & \text{if } R(B) \text{ is not closed.} \end{cases}$$

Proof By the proof of Theorem 2.2, we need to prove only that if $A \in \Phi_+(H)$ and $R(B)$ is closed and if $\exists\, C \in B(K, H)$ such that $M_C \in \Phi_+(H \oplus K)$ and $n(M_C) = n(A)$, then $n(B) \leq d(A)$. If $d(A) = \infty$, the result is clearly true. Then, we suppose that $d(A) = m < \infty$, then $n(B) < \infty$. Suppose $n(B) = n$ and let $\{e_1, e_2, \ldots, e_n\}$ be an orthonormal basis for $N(B)$. If $n > m$, let $Ce_i = \alpha_i + \beta_i$, where $\alpha_i \in R(A)$ and $\beta_i \in R(A)^\perp$. Then $\{Ce_i - \alpha_i\}$ $(i = 1, 2, \ldots, n)$ are linearly dependent. There exists $\{a_i\} \subseteq \mathbb{C}$ such that $a_j \neq 0$ for some j and $\sum_{i=1}^n a_iCe_i = \sum_{i=1}^n a_i\alpha_i \in CN(B) \cap R(A)$. Since $N(M_C) = N(A) \oplus \{0\}$, it follows that $CN(B) \cap R(A) = \{0\}$. Therefore $\sum_{i=1}^n a_iCe_i = 0$. We have that $C|_{N(B)}$ is injective. To see this, if not, there exists $y \in N(B)$ such that $y \neq 0$ and $Cy = 0$. Then $\binom{0}{y} \in N(M_C)$. It is in contradiction to the fact that $N(M_C) = N(A) \oplus \{0\}$. Therefore $\sum_{i=1}^n a_ie_i = 0$. It is a contradiction. Then $n(B) \leq d(A)$.

Corollary 2.7 *A 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is bounded below for some $C \in B(K, H)$ if and only if A is bounded below and*:

$$\begin{cases} n(B) \leq d(A), & \text{if } R(B) \text{ is closed;} \\ d(A) = \infty, & \text{if } R(B) \text{ is not closed.} \end{cases}$$

Corollary 2.8 *An 2×2 operator matrix $M_C = \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ is a lower semi-Fredholm operator for some $C \in B(K, H)$ and $d(M_C) = d(B)$ if and only if B is a lower semi-Fredholm operator* *and*:

$$\begin{cases} d(A) \leq n(B), & \textit{if } R(A) \textit{ is closed;} \\ n(B) = \infty, & \textit{if } R(A) \textit{ is not closed.} \end{cases}$$

The following corollary is immediate from Theorems 2.2 and 2.3:

Corollary 2.9 *For a given pair* (A, B) *of operators, we have*

$$\bigcap_{C \in B(K,H)} \sigma_{SF_+}(M_C) = \sigma_{SF_+}(A) \bigcup \{\lambda \in \mathbb{C} : \ R(B - \lambda I) \textit{ is not closed and } d(A - \lambda I) < \infty\}$$
$$\bigcup \{\lambda \in \mathbb{C} : \ R(B - \lambda I) \textit{ is closed and } n(B - \lambda I) = \infty, \ d(A - \lambda I) < \infty\}$$

and

$$\bigcap_{C \in B(K,H)} \sigma_{SF_-}(M_C) = \sigma_{SF_-}(B) \bigcup \{\lambda \in \mathbb{C} : \ R(A - \lambda I) \textit{ is not closed and } n(B - \lambda I) < \infty\}$$
$$\bigcup \{\lambda \in \mathbb{C} : \ R(A - \lambda I) \textit{ is closed and } d(A - \lambda I) = \infty, \ n(B - \lambda I) < \infty\}.$$

3 The Passage from $\sigma_{SF_\pm}(A) \cup \sigma_{SF_\pm}(B)$ $(\sigma_e(A) \cup \sigma_e(B))$ to $\sigma_{SF_\pm}(M_C)$ $(\sigma_e(M_C))$

In [4], it was shown that for every $C \in B(K, H)$, the passage from $\sigma_w(A) \cup \sigma_w(B)$ to $\sigma_w(M_C)$ is accomplished by removing certain open subsets of $\sigma_w(A) \cap \sigma_w(B)$ from the former, that is, there is the equality

$$\eta(\sigma_w(A) \bigcup \sigma_w(B)) = \eta(\sigma_w(M_C)),$$

where $\eta(\cdot)$ denotes the "polynomially-convex hull". More precisely,

$$\sigma_w(A) \bigcup \sigma_w(B) = \sigma_w(M_C) \bigcup \mathscr{G} \ ,$$

where $\mathscr{G}$ is the union of certain of the holes in $\sigma_w(M_C)$ which happen to be subsets of $\sigma_w(A) \cap \sigma_w(B)$. The passage from $\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B)$ $(\sigma_{SF_-}(A) \cup \sigma_{SF_-}(B))$ to $\sigma_{SF_+}(M_C)$ $(\sigma_{SF_-}(M_C))$ is more delicate.

Theorem 3.1 *For a given pair* (A, B) *of operators, we have that for every* $C \in B(K, H)$,

$$\eta(\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B)) = \eta(\sigma_{SF_+}(M_C)),$$

where $\eta(\cdot)$ *denotes the "polynomially-convex hull". More precisely,*

$$\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B) = \sigma_{SF_+}(M_C) \cup \mathscr{G},$$

where $\mathscr{G}$ *lies in certain holes in* $\sigma_{SF_+}(M_C)$, *which happen to be subsets of* $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$.

Proof First we claim that, for every $T \in B(H)$,

$$\eta(\sigma_{SF_+}(T)) = \eta(\sigma_w(T)). \tag{2}$$

Since $\sigma_{SF_+}(T) \subseteq \sigma_w(T)$, we need to prove $\partial\, \sigma_w(T) \subseteq \partial\, \sigma_{SF_+}(T)$. But since int $\sigma_{SF_+}(T) \subseteq$ int $\sigma_w(T)$, it suffices to show that $\partial \sigma_w(T) \subseteq \sigma_{SF_+}(T)$. Suppose $\lambda_0 \in \partial \sigma_w(T) \backslash \sigma_{SF_+}(T)$. By the perturbation theory of upper semi-Fredholm, there exists $\epsilon > 0$ such that $T - \lambda I \in \Phi_+(H)$ and $\mathrm{ind}(T - \lambda I) = \mathrm{ind}(T - \lambda_0 I)$ if $0 < |\lambda - \lambda_0| < \epsilon$. Since $\lambda_0 \in \partial \sigma_w(T)$, there exists λ_1 such that $0 < |\lambda_1 - \lambda_0| < \epsilon$ and $T - \lambda_1 I$ is Weyl. Then $T - \lambda_0 I$ is Weyl. It is in contradiction to the fact that $\lambda_0 \in \sigma_w(T)$ and hence $\partial\, \sigma_w(T) \subseteq \sigma_{SF_+}(T)$. This proves (2). Similarly, for every $T_1 \in B(H)$ and $T_2 \in B(K)$, $\eta(\sigma_{SF_+}(T_1) \cup \sigma_{SF_+}(T_2)) = \eta(\sigma_w(T_1) \cup \sigma_w(T_2))$. Then for each $C \in B(K, H)$,

$$\eta(\sigma_{SF_+}(M_C)) = \eta(\sigma_w(M_C)) = \eta(\sigma_w(A) \cup \sigma_w(B)) = \eta(\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B)). \tag{3}$$

Now suppose $\lambda \in (\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B)) \backslash \sigma_{SF_+}(M_C)$. Then $\lambda \in \sigma_{SF_+}(B) \backslash \sigma_{SF_+}(A)$. By Theorem 2.2, if $R(B - \lambda I)$ is closed, then $n(B - \lambda I) = \infty$ and hence $d(A - \lambda I) = \infty$. If instead $R(B - \lambda I)$ is not closed, then using Theorem 2.2 again, $d(A - \lambda I) = \infty$. Therefore $\lambda \in \sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$. (3) says that the passage from $\sigma_{SF_+}(M_C)$ to $\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B)$ is the filling in certain of the holes in $\sigma_{SF_+}(M_C)$. But since $\sigma_{SF_+}(A) \cup \sigma_{SF_+}(B) \backslash \sigma_{SF_+}(M_C)$ is contained in $\sigma_{SF_-}(A)\cap\sigma_{SF_+}(B)$, it follows that the filling in certain of the holes in $\sigma_{SF_+}(M_C)$ should occur in $\sigma_{SF_-}(A)\cap\sigma_{SF_+}(B)$. The proof is completed.

For lower semi-Fredholm spectrum and essential spectrum, we also have:

Theorem 3.2 *For a given pair (A,B) of operators, we have that for every $C\in B(K,H)$,*

$$\eta(\sigma_{SF_-}(A)\cup\sigma_{SF_-}(B))=\eta(\sigma_{SF_-}(M_C)),$$

where $\eta(\cdot)$ denotes the "polynomially-convex hull". More precisely,

$$\sigma_{SF_-}(A)\cup\sigma_{SF_-}(B)=\sigma_{SF_-}(M_C)\cup\mathscr{G},$$

where $\mathscr{G}$ lies in certain holes in $\sigma_{SF_-}(M_C)$, which happen to be subsets of $\sigma_{SF_-}(A)\cap\sigma_{SF_+}(B)$.

Theorem 3.3 *For a given pair (A,B) of operators, we have that for every $C\in B(K,H)$,*

$$\eta(\sigma_e(A)\cup\sigma_e(B))=\eta(\sigma_e(M_C)),$$

where $\eta(\cdot)$ denotes the "polynomially-convex hull". More precisely,

$$\sigma_e(A)\cup\sigma_e(B)=\sigma_e(M_C)\cup\mathscr{G},$$

where $\mathscr{G}$ lies in certain holes in $\sigma_e(M_C)$, which happen to be subsets of $\sigma_{SF_-}(A)\cap\sigma_{SF_+}(B)$.

Corollary 3.4 *If $\sigma_{SF_-}(A)\cap\sigma_{SF_+}(B)$ has no interior points, then for every $C\in B(K,H)$,*

$$\sigma_{SF_+}(A)\cup\sigma_{SF_+}(B)=\sigma_{SF_+}(M_C);\quad \sigma_{SF_-}(A)\cup\sigma_{SF_-}(B)=\sigma_{SF_-}(M_C);$$

and $\sigma_e(A)\cup\sigma_e(B)=\sigma_e(M_C)$.

4 Weyl's Theorem for 2×2 Upper Triangular Operator Matrices

Weyl [6] examined the spectra of all compact perturbations $A+K$ of a Hermitian operator A and discovered that $\lambda\in\sigma(A+K)$ for every compact operator K if and only if λ is not an isolated eigenvalue of finite multiplicity in $\sigma(A)$. Today this result is known as Weyl's theorem. Similar to Weyl's theorem, there is a-Weyl's theorem ([7, 8]).

It is well known that Weyl's theorem holds for $A\in B(H)$ if

$$\sigma(A)\backslash\sigma_w(A)=\pi_{00}(A);$$

and Browder's theorem holds for A if

$$\sigma_w(A)=\sigma_b(A).$$

Clearly, Weyl's theorem implies Browder's theorem.

Let $\Phi_+^-(H)$ be the class of all $A\in\Phi_+(H)$ with $\mathrm{ind}(A)\le 0$, and for any $A\in B(H)$, let

$$\sigma_{ea}(A)=\{\lambda\in\mathbb{C}: A-\lambda I \text{ is not in } \Phi_+^-(H)\}$$

and

$$\sigma_{ab}(A)=\{\lambda\in\mathbb{C}: A-\lambda I \text{ is not an upper semi-Fredholm operator with finite ascent}\}.$$

We call $\sigma_{ea}(A)$ the essential approximate point spectrum of A and $\sigma_{ab}(A)$ the Browder essential approximate point spectrum of A.

Similarly, we say that a-Weyl's theorem holds for A if there is equality

$$\sigma_a(A)\backslash\sigma_{ea}(A)=\pi_{00}^a(A);$$

and that a-Browder's theorem holds for A if there is equality

$$\sigma_{ea}(A)=\sigma_{ab}(A).$$

It is known ([7, 8]) that if $A\in B(H)$, then we have

$$\begin{aligned}\text{a-Weyl's theorem} &\Longrightarrow \text{a-Browder's theorem and Weyl's theorem}\\ &\Longrightarrow \text{Browder's theorem.}\end{aligned}$$

Weyl's theorem may or may not hold for a direct sum of operators for which Weyl's theorem holds. Thus Weyl's theorem may fail for upper triangular operator matrices, so does a-Weyl's theorem. Weyl's theorem for upper triangular operator matrices is more delicate in comparison with the diagonal matrices. In this section, we consider this question: If Weyl's (a-Weyl's) theorem holds for $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$, when does it hold for $\left(\begin{smallmatrix}A&C\\0&B\end{smallmatrix}\right)$? We begin with:

Lemma 4.1 *For a given pair (A,B) of operators, if both A and B have finite ascents, then for every $C\in B(K,H)$, M_C has finite ascent.*

Proof Suppose $\alpha(A)=p$ and $\alpha(B)=q$, let $n=\max\{p,q\}$. For every $C\in B(K,H)$, if we have $N(M_C^{2n+1})=N(M_C^{2n})$, we get the result. So we need to prove only $N(M_C^{2n+1})\subseteq N(M_C^{2n})$.

If $u_0\in N(M_C^{2n+1})$, supposing $u_0=(x_0,y_0)$, then

$$\begin{aligned}0&=M_C^{2n+1}(x_0,y_0)\\&=(A^{2n+1}x_0+A^{2n}Cy_0+A^{2n-1}CBy_0+\cdots+A^nCB^ny_0+\cdots+CB^{2n}y_0,\ B^{2n+1}y_0),\end{aligned}$$

then $B^{2n+1}y_0=0$ and

$$A^{2n+1}x_0+A^{2n}Cy_0+A^{2n-1}CBy_0+\cdots+A^nCB^ny_0+\cdots+CB^{2n}y_0=0.$$

So $y_0\in N(B^{2n+1})=N(B^n)$, thus

$$A^{2n+1}x_0+A^{2n}Cy_0+A^{2n-1}CBy_0+\cdots+A^{n+1}CB^{n-1}y_0=0,$$

that is,

$$A^{n+1}[A^nx_0+A^{n-1}Cy_o+A^{n-2}CBy_0+\cdots+CB^{n-1}y_0]=0,$$

and hence

$$A^nx_0+A^{n-1}Cy_0+A^{n-2}CBy_0+\cdots+CB^{n-1}y_0\in N(A^{n+1})=N(A^n).$$

Then

$$A^{2n}x_0+A^{2n-1}Cy_0+A^{2n-2}CBy_0+\cdots+A^nCB^{n-1}y_0=0.$$

Now we get that

$$(A^{2n}x_0+A^{2n-1}Cy_0+\cdots+A^nCB^{n-1}y_0+A^{n-1}CB^ny_0+\cdots+CB^{2n-1}y_0,\ B^{2n}y_0)=0,$$

that is, $M_C^{2n}u_0=0$ and hence $u_0\in N(M_C^{2n})$. So $N(M_C^{2n+1})=N(M_C^{2n})$, and hence M_C has finite ascent.

Theorem 4.2 *If $\sigma_{SF_-}(A)\cap\sigma_{SF_+}(B)$ has no interior points, then for every $C\in B(K,H)$:*

(a) *Browder's theorem holds for $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)\Longrightarrow$ Browder's theorem holds for $\left(\begin{smallmatrix}A&C\\0&B\end{smallmatrix}\right)$;*

(b) *a-Browder's theorem holds for $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)\Longrightarrow$ a-Browder's theorem holds for $\left(\begin{smallmatrix}A&C\\0&B\end{smallmatrix}\right)$.*

Proof (a) We need to prove that $\sigma_w(M_C)=\sigma_b(M_C)$ for every $C\in B(K,H)$. Since $\sigma_w(M_C)\subseteq\sigma_b(M_C)$, we need to prove only $\sigma_b(M_C)\subseteq\sigma_w(M_C)$. Suppose that $M_C-\lambda_0I$ is Weyl. Then $A-\lambda_0I\in\Phi_+(H)$ and $B-\lambda_0I\in\Phi_-(K)$ and $A-\lambda_0I$ is Fredholm $\Longleftrightarrow B-\lambda_0I$ is Fredholm. Corollary 3.4 asserts that $\sigma_{SF_+}(A)\cup\sigma_{SF_+}(B)=\sigma_{SF_+}(M_C)$. Since λ_0 is not in $\sigma_{SF_+}(M_C)$, it follows that λ_0 is not in $\sigma_{SF_+}(B)$. Then $B-\lambda_0I$ is Fredholm and hence $A-\lambda_0I$ is Fredholm. Thus by $\mathrm{ind}\big(\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)-\lambda_0I\big)=\mathrm{ind}(M_C-\lambda_0I)=0$, we get that $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)-\lambda_0I$ is Weyl. Browder's theorem holds for $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$, then $\alpha(A-\lambda_0I)<\infty$ and $\alpha(B-\lambda_0I)<\infty$. Therefore $\alpha(M_C-\lambda_0I)<\infty$ (Lemma 4.1). [9, Theorem 4.5] asserts that $M_C-\lambda_0I$ is Browder. Then Browder's theorem holds for M_C for every $C\in B(K,H)$.

(b) Since $\sigma_{ea}(M_C)\subseteq\sigma_{ab}(M_C)$, we need to prove only that $\sigma_{ab}(M_C)\subseteq\sigma_{ea}(M_C)$. Suppose $M_C-\lambda_0I\in\Phi_+^-(H\oplus K)$. Then $A-\lambda_0I\in\Phi_+(H)$ and λ_0 is not in $\sigma_{SF_+}(M_C)=\sigma_{SF_+}(A)\cup\sigma_{SF_+}(B)$. Therefore $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)-\lambda_0I\in\Phi_+(H\oplus K)$ and $\mathrm{ind}\big(\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)-\lambda_0I\big)=\mathrm{ind}(M_C-\lambda_0I)\le0$, which means that λ_0 is not in $\sigma_{ea}\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$. a-Browder's theorem holds for $\left(\begin{smallmatrix}A&0\\0&B\end{smallmatrix}\right)$, then $\alpha(A-\lambda_0I)<\infty$ and $\alpha(B-\lambda_0I)<\infty$, hence $\alpha(M_C-\lambda_0I)<\infty$ (Lemma 4.1). It is shown that λ_0 is not in $\sigma_{ab}(M_C)$. Then $\sigma_{ea}(M_C)=\sigma_{ab}(M_C)$ for every $C\in B(K,H)$, and hence a-Browder's theorem holds for M_C.

$A\in B(H)$ is called approximate-isoloid (abbrev. a-isoloid) if every isolated point of $\sigma_a(A)$ is an eigenvalue of A, and A is called isoloid if every isolated point of $\sigma(A)$ is an eigenvalue of A.

Remark 4.3 Theorem 4.2 may fail for "a-Weyl's theorem" even with the additional assumption that a-Weyl's theorem holds for A and B and both A and B are a-isoloid. To see this, let

$A,\ B,\ C \in B(\ell_2)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots),$$
$$B(x_1, x_2, x_3, \ldots) = (0, x_2, 0, x_4, 0, x_6, \ldots),$$
$$C(x_1, x_2, x_3, \ldots) = \left(0, 0, 0, 0, \frac{1}{3}x_3, 0, \frac{1}{5}x_5, \ldots\right).$$

Then

$$\sigma_a(A) = \sigma_{ea}(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\},\ \sigma_{SF_-}(A) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\},\ \pi^a_{00}(A) = \emptyset,$$
$$\sigma_a(B) = \sigma_{ea}(B) = \sigma_{SF_+}(B) = \{0, 1\},\ \pi^a_{00}(B) = \emptyset,$$

which says that a-Weyl's theorem holds for A and B, both A and B are a-isoloid and $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$ has no interior points. Also a straightforward calculation shows that

$$\sigma_a\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_{ea}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_a\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \sigma_{ea}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \{\lambda \in \mathbb{C} : |\lambda| = 1\} \cup \{0\},$$
$$\pi^a_{00}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset, \quad \pi^a_{00}\begin{pmatrix} A & C \\ 0 & B \end{pmatrix} = \{0\}.$$

Then a-Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, but fails for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$.

But for Weyl's theorem, we have:

Theorem 4.4 *If $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$ has no interior points and if A is an isoloid operator for which Weyl's theorem holds, then for every $C \in B(K, H)$,*

Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \Longrightarrow$ *Weyl's theorem holds for* $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$.

Proof Theorem 4.2 gives that $\sigma(M_C) \backslash \sigma_w(M_C) \subseteq \pi_{00}(M_C)$. For the reverse inclusion, suppose that $\lambda_0 \in \pi_{00}(M_C)$. Then there exists $\epsilon > 0$ such that $M_C - \lambda I$ is invertible and hence $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective if $0 < |\lambda - \lambda_0| < \epsilon$. Since $\sigma_{SF_+}(M_C) = \sigma_{SF_+}(A) \cup \sigma_{SF_+}(B)$ and λ is not in $\sigma_{SF_+}(M_C)$, it follows that $B - \lambda I$ is Fredholm and hence $A - \lambda I$ is Fredholm. Then $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) - \lambda I$ is Fredholm with $\mathrm{ind}\big(\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) - \lambda I\big) = \mathrm{ind}(M_C - \lambda I) = 0$, that is, $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) - \lambda I$ is Weyl. Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$, then $A - \lambda I$ and $B - \lambda I$ are Browder if $0 < |\lambda - \lambda_0| < \epsilon$. Thus $A - \lambda I$ and $B - \lambda I$ are invertible because $A - \lambda I$ is bounded below and $B - \lambda I$ is surjective. Now we have that $\lambda_0 \in \mathrm{iso}\, \sigma\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$. The following proof is the same as the proof in Theorem 2.4 in [5]:

Remark 4.5 Theorem 4.4 in this paper is not compatible with Theorem 2.4 in [5]. For example:

(a) Let $A, B \in B(\ell_2)$ be defined by

$$A(x_1, x_2, x_3, \ldots) = (x_2, x_4, x_6, \ldots), \quad B(x_1, x_2, x_3, \ldots) = (0, x_1, 0, x_2, 0, x_3, \ldots).$$

Then

$$\sigma(A) = \sigma_w(A) = \sigma_e(A) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\},\ \sigma_{SF_-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = 1\},\ \pi_{00}(A) = \emptyset,$$
$$\sigma_{SF_+}(B) = \{\lambda \in \mathbb{C} : |\lambda| = 1\},\ \sigma_e(B) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\},$$
$$\sigma\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \{\lambda \in \mathbb{C} : |\lambda| \le 1\},\ \pi_{00}\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset.$$

We have:

(I) $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B)$ has no interior points;

(II) Both $SP(A)$ and $SP(B)$ have pseudoholes;

(III) A is isoloid and Weyl's theorem holds for A;

(IV) Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

Then using Theorem 4.4 in this paper, Weyl's theorem holds for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ for every $C \in B(\ell_2)$. But using Theorem 2.4 in [5], we don't know whether Weyl's theorem holds for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ for every $C \in B(\ell_2)$.

(b) Let $T_1, T_2 \in B(\ell_2)$ be defined by

$$T_1(x_1, x_2, x_3, \dots) = (0, x_1, 0, x_2, 0, x_3, \dots), \quad T_2(x_1, x_2, x_3, \dots) = (x_2, x_4, x_6, \dots)$$

and let

$$A = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ and } B = T_2.$$

Then

$$\sigma(A) = \sigma_w(A) = \sigma_e(A) = \sigma_{SF_+}(A) = \sigma_{SF_-}(A) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}, \ \pi_{00}(A) = \emptyset,$$

$$\sigma \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \sigma_w \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}, \ \pi_{00} \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \emptyset.$$

This means that

(I) $\sigma_{SF_-}(A) \cap \sigma_{SF_+}(B) = \{\lambda \in \mathbb{C} : |\lambda| \le 1\}$ has interior points;

(II) $SP(A)$ has no pseudoholes;

(III) A is isoloid and Weyl's theorem holds for A;

(IV) Weyl's theorem holds for $\left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right)$.

Then by Theorem 2.4 in [5], Weyl's theorem holds for $\left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right)$ for every $C \in B(K, H)$. But using Theorem 4.4 in this paper, we don't know whether Weyl's theorem holds for M_C.

In spite of Remark 4.3, similarly to the proof of Theorem 4.4, we have:

Theorem 4.6 *Suppose that $\sigma_{SF_-}(A)$ has no interior points and $\sigma_{ab}(B) = \sigma_a(B)$. If A is an a-isoloid operator for which a-Weyl's theorem holds, then for every $C \in B(K, H)$,*

$$a\text{-}Weyl\text{'s theorem holds for } \left(\begin{smallmatrix} A & 0 \\ 0 & B \end{smallmatrix}\right) \Longrightarrow a\text{-}Weyl\text{'s theorem holds for } \left(\begin{smallmatrix} A & C \\ 0 & B \end{smallmatrix}\right).$$

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