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Hyperbolic L_2-modules with Reproducing Kernels

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Abstract In this paper, the Dirac operator on the Klein model for the hyperbolic space is considered. A function space containing L_2-functions on the sphere S^{m-1} in $\mathbb{R}^m$, which are boundary values of solutions for this operator, is defined, and it is proved that this gives rise to a Hilbert module with a reproducing kernel.

Keywords Hyperbolic space, Clifford analysis, Cauchy transform, Lie sphere
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1 Introduction

In this paper Clifford analysis techniques are used to construct subspaces of $L_2(S^{m-1})$, the space of square integrable functions on the sphere S^{m-1}, which contain boundary values of solutions for the Dirac operator on the hyperbolic unit ball. In Section 2 several models for the hyperbolic unit ball will be given. The Dirac operator is at the very heart of Clifford analysis, offering a direct and elegant generalization of the theory of holomorphic functions in the complex plane. Standard reference books on the theory of Clifford analysis on the flat Euclidean space $\mathbb{R}^m$ are [1], [2] and [3], and a nice overview of the most basic results is given in [4]. In Section 3 we recall the basic notions and concepts of Clifford analysis which will be used in the course of this paper. In Section 4 we then define the Dirac operator on the Klein model for the hyperbolic unit ball and we state the main results from the function theory for this operator. In Section 5 a function space containing boundary values of hyperbolic monogenics is introduced and by means of some results from the theory of Clifford analysis on the Lie sphere, it is proved that this function space has a reproducing kernel.

2 The Hyperbolic Model

Consider the real orthogonal space $\mathbb{R}^{1,m}$, i.e. the *flat Minkowski space-time*, with orthonormal basis $\{\epsilon, e_1, \ldots, e_m\}$ and endowed with the quadratic form

$$Q_{1,m}(T,\underline{X}) = T^2 - \sum_{j=1}^{m} X_j^2 = T^2 - |\underline{X}|^2.$$

Space-time vectors are denoted by $\epsilon T + \underline{X}$, where we prefer to make a clear distinction between the temporal coordinate T and the spatial coordinates X_j $(1 \leq j \leq m)$. The nullcone NC is defined as the set of space-time vectors for which $Q_{1,m}(T,\underline{X}) = 0$, and separates the time-like region $(Q_{1,m}(T,\underline{X}) > 0)$ from the space-like region $(Q_{1,m}(T,\underline{X}) < 0)$. The time-like region is the union of the future cone FC $(T > 0)$ and the past cone PC $(T < 0)$.

The hyperbolic unit ball H_+ is then defined as the following subset of FC:

$$H_+ = \{\epsilon T + \underline{X} \in FC : Q_{1,m}(T,\underline{X}) = 1\}.$$

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This m-dimensional surface embedded in flat Minkowski space-time $\mathbb{R}^{1,m}$ is to be interpreted as the hyperbolic analogue of the unit sphere S^m embedded in the flat Euclidean space $\mathbb{R}^{m+1}$. A *projective model* for the hyperbolic unit ball is obtained by considering the manifold of rays

$$\mathrm{ray}(FC) = \left\{\lambda(\epsilon T + \underline{X}) : \epsilon T + \underline{X} \in FC, \lambda \in R_+\right\}$$

inside the future cone, see e.g. [5] and [6], [7]. Other models are then readily obtained by intersecting this manifold $\mathrm{ray}(FC)$ with an arbitrary surface $\Sigma \subset FC$ such that each ray intersects Σ at a unique point.

The Klein model is obtained by intersecting $\mathrm{ray}(FC)$ with the hyperplane $\Pi \leftrightarrow T = 1$, realizing the hyperbolic unit ball inside $B_m(1) \subset \mathbb{R}^m$. Provided with the so-called Cayley–Klein–Hilbert metric ds_K^2, one obtains a metric model $(B_m(1), ds_K^2)$ for the hyperbolic unit ball which is, however, *not* conformal. This is a serious disadvantage, but it is compensated for by the fact that the straight lines in this model are restrictions to $B_m(1)$ of straight lines with respect to the standard Euclidean metric on $B_m(1)$, i.e. straight lines in the Klein model for the hyperbolic unit ball are chords in $B_m(1)$. In the case $m = 2$ one obtains the classical Klein model for the hyperbolic plane, whose metric ds_K^2, in coordinates (x, y) on $\mathbb{R}^2$, is given by

$$ds_K^2 = \frac{(1-y^2)dx^2}{(1-r^2)^2} + \frac{xydxdy}{(1-r^2)^2} + \frac{(1-x^2)dy^2}{(1-r^2)^2}.$$

3 The Clifford Setting

As we use the Klein model for the hyperbolic unit ball realized in the flat Euclidean space, we need both the Clifford algebra $\mathbb{R}_{1,m}$ associated with the flat Minkowksi space-time and the Clifford algebra $\mathbb{R}_m$. In Section 5 we will then need some results concerning Clifford analysis on the Lie sphere too, whence we consider three subsections:

3.1 Clifford Analysis on Flat Euclidean Space

Let $\{e_1, \ldots, e_m\}$ be an orthonormal basis for $\mathbb{R}^m$ endowed with the standard Euclidean inner product $\langle \underline{x}, \underline{y} \rangle = \sum_j x_j y_j$. The Clifford algebra $\mathbb{R}_m$ is then defined as the 2^m-dimensional real associative, but non-commutative, algebra generated by the basis for $\mathbb{R}^m$ with multiplication rule: $e_j e_j + e_j e_i = -2\delta_{ij}$. An element of $\mathbb{R}_m$ is called a Clifford number and has the form $a = \sum_{A \subset M} a_A e_A$, with $a_A \in \mathbb{R}$ and where $A = \{i_1, \ldots, i_k\}$, $i_1 < \cdots < i_k$ is a subset of $M = \{1, \ldots, m\}$ such that $e_A = e_{i_1} \cdots e_{i_k}$. If A has k elements, e_A is called a k-vector. Denoting the projection of a Clifford number a on its k-vector part as $[a]_k$, we get $a = \sum_{k=0}^m [a]_k$, with $[a]_k \in \mathbb{R}_m^{(k)}$. The *even subalgebra* is defined as the subspace $\mathbb{R}_m^{(+)} = \sum_{k \text{ even}} \mathbb{R}_m^{(k)}$ of $\mathbb{R}_m$.

Vectors in $\mathbb{R}^m$ are identified with 1-vectors in $\mathbb{R}_m$. Note that for $\underline{x}$ and $\underline{y} \in \mathbb{R}^m$, the Clifford product $\underline{x}\underline{y} = \underline{x} \cdot \underline{y} + \underline{x} \wedge \underline{y}$ incorporates both the *inner product* $\underline{x} \cdot \underline{y} = -\langle \underline{x}, \underline{y} \rangle = -\sum_{j=1}^m x_j y_j$ and the *outer product* $\underline{x} \wedge \underline{y} = \sum_{i<j} e_{ij}(x_i y_j - x_j y_i)$. The conjugation (bar-map) on $\mathbb{R}_m$ is defined as the anti-automorphism sending $e_j \mapsto \overline{e}_j = -e_j$, with $\overline{ab} = \overline{b}\overline{a}$. Its extension to the complexified Clifford algebra $\mathbb{C}_m = \mathbb{R}_m \otimes \mathbb{C}$ is the Hermitian conjugation $a \mapsto a^+$, given by the tensor product of the bar-map on $\mathbb{R}_m$ and the classical complex conjugation.

The Clifford group $\Gamma(m)$ is the subgroup of $\mathbb{R}_m$ generated by the non-zero vectors; the Pin group $\mathrm{Pin}(m)$ is the subgroup of $\Gamma(m)$ consisting of products of unit vectors in $\mathbb{R}^m$ and the Spin group $\mathrm{Spin}(m)$ is the subgroup of $\mathrm{Pin}(m)$ consisting of products of an *even* number of unit vectors. For an element $s \in \mathrm{Pin}(m)$ the map $\chi(s) : \mathbb{R}^m \mapsto \mathbb{R}^m : \underline{x} \mapsto s\underline{x}\overline{s}$ induces an orthogonal transformation on $\mathbb{R}^m$. In this way, $\mathrm{Pin}(m)$ defines a double covering of the group $\mathrm{O}(m)$ whereas the Spin group defines a double covering of $\mathrm{SO}(m)$.

The Dirac operator on $\mathbb{R}^m$ is defined as the vector derivative $\underline{\partial} = \sum_j e_j \partial_j$, which is a first-order $\mathrm{Spin}(m)$-invariant differential operator factorizing the Laplacian Δ_m on $\mathbb{R}^m$: $\underline{\partial}^2 = -\Delta_m$. Let Ω be an open subset of $\mathbb{R}^m$ and let $f : \Omega \mapsto \mathbb{R}_m$ be an element of $C^1(\Omega)$. If $\underline{\partial} f = 0$ in Ω, f is called *monogenic* on Ω. It is clear that monogenic functions in Ω form a subclass of harmonic functions in Ω. In polar coordinates the Dirac operator admits the following decomposition:

$\underline{\partial} = \underline{\xi}(\partial_r + \frac{1}{r}\Gamma)$, where $\underline{x} = r\underline{\xi}$ and $\Gamma = -\underline{x} \wedge \underline{\partial}$ is the spherical Dirac operator on S^{m-1}. This operator is strongly related to the Dirac operator on the sphere, see e.g. [8], [9] and [10]. In terms of the so-called momentum operators $L_{ij} = x_i\partial_{x_j} - x_j\partial_{x_i}$ it is given by $\Gamma = -\sum_{i<j} e_{ij}L_{ij}$.

The restriction $P_k(\underline{\xi})$ to S^{m-1} of a k-homogeneous monogenic polynomial $P_k(\underline{x})$ is called an *inner spherical monogenic* of order k. It is an eigenfunction of Γ satisfying $\Gamma P_k = -kP_k$. The restriction $Q_k(\underline{\xi})$ of a homogeneous monogenic function $Q_k(\underline{x})$ of degree $(1-k-m)$ on $\mathbb{R}^m \setminus \{\underline{0}\}$ is an *outer spherical monogenic* of order k, satisfying $\Gamma Q_k = (k+m-1)Q_k$. The (right Clifford) modules containing these functions are, respectively, denoted as $M_+(k)$ and $M_-(k)$. Inner and outer spherical monogenics are related as follows: $P_k(\underline{\xi}) \in M_+(k) \Rightarrow \underline{\xi}P_k(\underline{\xi}) \in M_-(k)$ and vice versa. Each function $f \in L_2(S^{m-1})$ can be decomposed as $f(\underline{\xi}) = \sum_{k=0}^{\infty} P(k)[f](\underline{\xi}) + Q(k)[f](\underline{\xi})$ where the series converges in L_2-sense. The projections on $M_+(k)$ and $M_-(k)$ are given by

$$P(k)[f](\underline{\omega}) = \frac{1}{A_m}\int_{S^{m-1}} \left[C_k^{\frac{m}{2}}(\langle\underline{\xi},\underline{\omega}\rangle) + \underline{\omega}\underline{\xi}C_{k-1}^{\frac{m}{2}}(\langle\underline{\xi},\underline{\omega}\rangle)\right] f(\underline{\xi})dS(\underline{\xi})$$

and

$$Q(k)[f](\underline{\omega}) = -\frac{1}{A_m}\underline{\omega}\int_{S^{m-1}} \left[C_k^{\frac{m}{2}}(\langle\underline{\xi},\underline{\omega}\rangle) + \underline{\omega}\underline{\xi}C_{k-1}^{\frac{m}{2}}(\langle\underline{\xi},\underline{\omega}\rangle)\right] \underline{\xi}f(\underline{\xi})dS(\underline{\xi}).$$

The fundamental solution for the Dirac operator is given by the so-called Cauchy kernel $E(\underline{x})$, defined as $E(\underline{x}) = \frac{\underline{x}}{|\underline{x}|^m}$, $\underline{x} \in \mathbb{R}^m$, satisfying $\underline{\partial}E(\underline{x}) = -\delta(\underline{x}) = E(\underline{x})\underline{\partial}$. Due to translational invariance of the Dirac operator on $\mathbb{R}^m$, we also have $\underline{\partial}_x E(\underline{x}-\underline{y}) = -\delta(\underline{x}-\underline{y})$.

3.2 Clifford Analysis on Flat Minkowski Space-time

The Clifford algebra $\mathbb{R}_{1,m}$ is generated by $\{\epsilon, e_1, \ldots, e_m\}$ with the multiplication rules $\epsilon e_i + e_i\epsilon = 0$, $e_ie_j + e_je_i = -2\delta_{ij}$ and $\epsilon^2 = 1$. For the definitions of Clifford numbers and k-vectors in $\mathbb{R}_{1,m}$ we refer to the flat Euclidean space. For two space-time vectors $\epsilon T + \underline{X}$ and $\epsilon S + \underline{Y}$, the Clifford product reduces to $(\epsilon T + \underline{X})(\epsilon S + \underline{Y}) = (\epsilon T + \underline{X}) \cdot (\epsilon S + \underline{Y}) + (\epsilon T + \underline{X}) \wedge (\epsilon S + \underline{Y})$, where the inner product is given by $(\epsilon T + \underline{X}) \cdot (\epsilon S + \underline{Y}) = ST - \langle\underline{X},\underline{Y}\rangle$ and the outer product by $(\epsilon T + \underline{X}) \wedge (\epsilon S + \underline{Y}) = S\underline{X}\epsilon - T\underline{Y}\epsilon + \underline{X} \wedge \underline{Y}$. In order to define the conjugation on $\mathbb{R}_{1,m}$ it suffices to note that $\overline{\epsilon} = -\epsilon$ and $\overline{a\epsilon} = -\epsilon\overline{a}$ for all $a \in \mathbb{R}_m$. Defining the so-called main involution by $\widetilde{\epsilon} = -\epsilon$, $\widetilde{e}_j = -e_j$ and $\widetilde{ab} = \widetilde{a}\widetilde{b}$ for all $a, b \in \mathbb{R}_{1,m}$, we are able to define the most important subgroups of $\mathbb{R}_{1,m}$: the Clifford group $\Gamma(1,m)$ is the set of invertible elements $a \in \mathbb{R}_{1,m}$ such that $a(\epsilon T + \underline{X})\widetilde{a}^{-1} \in \mathbb{R}^{1,m}$ for all $\epsilon T + \underline{X}$ in $\mathbb{R}^{1,m}$, the Pin group $\mathrm{Pin}(1,m)$ is the quotient group $\Gamma(1,m)/\mathbb{R}^+$ and the Spin group $\mathrm{Spin}(1,m) = \mathrm{Pin}(1,m) \cap \mathbb{R}_{1,m}^{(+)}$. The Pin group defines a double covering of $\mathrm{O}(1,m)$, whereas $\mathrm{Spin}(1,m)$ defines a double covering of $\mathrm{SO}(1,m)$.

The Dirac operator on the flat Minkowski space-time $\mathbb{R}^{1,m}$ is then defined as the vector derivative $\partial_X = \epsilon\partial_T - \partial_{\underline{X}}$, with $\partial_{\underline{X}} = \sum_j e_j\partial_j$, which is a $\mathrm{Spin}(1,m)$-invariant first-order differential operator factorizing the wave-operator on $\mathbb{R}^{1,m}$. A function $F(T,\underline{X})$ satisfying $\partial_X F = 0$ on an open subset $\Omega \subset \mathbb{R}^{1,m}$ is monogenic with respect to the Dirac operator on $\mathbb{R}^{1,m}$. However, such a function is *not* hyperbolic monogenic. A *hyperbolic monogenic* function is a monogenic function with respect to the operator ∂_X *defined on the hyperbolic unit ball*, which–in view of the fact that the true model for the hyperbolic unit ball is projective–means that it must be defined on the manifold of rays. This can be done by considering the homogeneous Clifford line bundle $\mathbb{R}_{1,m;\alpha} = \left\{ ((T,\underline{X}), a) \in \mathbb{R}_0^{1,m} \times \mathbb{R}_{1,m} \right\} / \sim$ where $\alpha \in \mathbb{C}$ and

$$((T,\underline{X}), a) \sim (\lambda(T,\underline{X}), \lambda^\alpha a)\,, \quad \lambda \in \mathbb{R}_+.$$

The Dirac operator on the hyperbolic unit ball is then defined as the Dirac operator ∂_X on the flat Minkowski space-time $\mathbb{R}^{1,m}$ acting on sections of this bundle, i.e. acting on α-homogeneous functions (α being an arbitrary complex number).

3.3 Clifford Analysis on the Lie Sphere

The Lie ball $LB_m(1)$ in $\mathbb{C}^m$ is defined by $LB_m(1) = \{z \in \mathbb{C}^m : L(\underline{z}) < 1\}$, where $L(\underline{z})$ is the Lie norm of $\underline{z} = (z_1, \ldots, z_m) = \underline{x} + i\underline{y}$, given by $L(\underline{z})^2 = |\underline{z}|^2 + (|\underline{z}|^4 - |\underline{z}^2|^2)^{\frac{1}{2}}$. Here, $|\underline{z}|^2$ stands for $\sum_j |z_j|^2$ and $\underline{z}^2$ for $\sum_j z_j^2$. The boundary $\partial LB_m(1)$ of the Lie ball has a part in common with the complex unit ball, viz. the elements $\underline{z} = \underline{x} + i\underline{y} \in \mathbb{C}^m$ such that $L(\underline{z}) = 1$, with $\underline{x}$ and $\underline{y}$ linearly dependent. This particular subset of the Lie ball is the so-called Lie sphere, which can be represented by $LS^{m-1} = \{e^{it}\underline{\omega}\ \ t \in \mathbb{R},\ \underline{\omega} \in S^{m-1}\}$. The Lie sphere can also be defined as $S^1 \times S^{m-1}/\sim$, where the equivalence relation is given by $(e^{it}, \underline{\omega}) \sim (-e^{it}, -\underline{\omega})$, whence functions on the Lie sphere can always be denoted by $f(e^{it}, \underline{\omega}) = f(e^{it}\underline{\omega})$. The importance of the Lie ball lies in the following theorem (see reference [11]):

Theorem (Siciak)

1 *If a series $\sum_k R_k(\underline{x})$ of homogeneous polynomials converges normally in $B_m(1)$, its complexification $\sum_k R_k(\underline{z})$ will converge normally in the Lie ball, and hence represent a holomorphic function there.*

2 *The Lie ball is the largest area where this is valid in general, i.e. there exists a harmonic function $h(\underline{x})$ such that the complexification $\sum_k S_k(\underline{z})$ of its development into spherical harmonics $S_k(\underline{x})$ can not be extended holomorphically beyond the Lie ball.*

Let us then consider $\mathbb{C}_m$-valued functions $f(\underline{z})$ on $\mathbb{C}^m$. Relevant operators on the Lie sphere are the Gamma operator Γ and the Euler operator $-i\partial_t$ and the simultaneous eigenfunctions of these operators are given by $(e^{it}\underline{\omega})^l P_k(e^{it}\underline{\omega})$, $l \in \mathbb{Z}$ and $P_k \in M_+(k)$. These functions are *spherical monogenics of order (k,l) on LS^{m-1}*, belonging to $\mathscr{M}_{k,l}$. They are the restrictions to LS^{m-1} of complex extensions of the Clifford monomials $\underline{x}^l P_k(\underline{x})$, viz. the simultaneous eigenfunctions of Γ and $r\partial_r$ on $\mathbb{R}^m$.

Defining the Hilbert module $L_2(LS^{m-1})$ of square integrable functions on the Lie sphere as $L_2(LS^{m-1}) = \{f(e^{it}\underline{\omega}) : \|f\|_{L_2(LS^{m-1})} < \infty\}$, where the Lie norm is given by $\|f\|^2_{L_2(LS^{m-1})} = [(f,f)]_0$, with

$$(f,g) \;=\; \frac{1}{\pi A_m}\int_0^\pi \int_{S^{m-1}} f(e^{it}\underline{\omega})^+ g(e^{it}\underline{\omega}) dS(\underline{\omega}) dt,$$

we have the orthogonal decomposition $L_2(LS^{m-1}) = \sum_{k=0}^\infty \sum_{l\in\mathbb{Z}} \mathscr{M}_{k,l}$.

Putting $\theta = \langle \underline{\omega}, \underline{\xi}\rangle$ for $\underline{\omega}$ and $\underline{\xi} \in S^{m-1}$, we have, for $f \in L_2(LS^{m-1})$

$$f(e^{it}\underline{\omega}) \;=\; \sum_{k=0}^\infty \sum_{l\in\mathbb{Z}} (e^{it}\underline{\omega})^l e^{ikt} P_{k,l} f(\underline{\omega})$$

with $P_{k,l}f(\underline{\omega})$ given by the integral

$$\frac{1}{\pi A_m}\int_0^\pi \int_{S^{m-1}} e^{-ikt}\{C_k^{\frac{m}{2}}(\theta) + \underline{\omega}\underline{\xi} C_{k-1}^{\frac{m}{2}}(\theta)\}(e^{it}\underline{\xi})^{-l} f(e^{it}\underline{\xi}) dS(\underline{\xi}) dt.$$

This decomposition refines the decomposition of functions $f \in L_2(LS^{m-1})$ into spherical harmonics on the Lie sphere (see e.g. references [12] and [13]).

Denoting the set of holomorphic functions on the Lie ball by $\mathscr{O}(LB_m(1))$, we then define the space $L_2^+(LS^{m-1})$ as the following Hardy-type space:

$$L_2^+(LS^{m-1}) = \left\{ f \in \mathscr{O}(LB_m(1)) : \lim_{r\to 1-} \int_0^\pi \int_{S^{m-1}} |f(re^{it}\underline{\omega})|^2 dS(\underline{\omega}) dt < \infty \right\}.$$

As was pointed out in reference [13], this module is a Hilbert module with reproducing kernel; the so-called Cauchy–Hua kernel $H(\underline{z}, \underline{w})$. This means that a function $f \in L_2^+(LS^{m-1})$ can be represented as

$$f(\underline{z}) = \frac{1}{\pi A_m}\int_0^\pi \int_{S^{m-1}} H^+(\underline{z}, e^{it}\underline{\omega}) f(e^{it}\underline{\omega}) dS(\underline{\omega}) dt,$$

where the Cauchy–Hua kernel is defined by

$$H(\underline{z}, e^{it}\underline{\omega}) = \frac{1}{\big(-(\underline{\omega} - e^{-it}\underline{z})^2\big)^{\frac{m}{2}}}.$$

The module $L_2^+(LS^{m-1})$ can also be defined as

$$L_2^+(LS^{m-1}) = \left\{ f \in L_2(LS^{m-1}) : f = \sum_{k=0}^\infty \sum_{l=0}^\infty (e^{it}\underline{\omega})^l P_{k,l} f(e^{it}\underline{\omega}) \right\},$$

defining $L_2^+(LS^{m-1})$ as a submodule of $L_2(LS^{m-1})$ containing boundary values of holomorphic functions in the Lie ball.

We end this section with a brief description of a technique to construct a reproducing kernel for a Hilbert module containing Clifford-algebra-valued nullsolutions for certain Clifford differential operators $P(\underline{x}, \underline{\partial})$ on the unit ball $B_m(1)$ with polynomial coefficients, the so-called *operators of Frobenius type* satisfying the conditions of the following theorem (see [14] and [15]):

Definition *A differential operator $P(\underline{x}, \underline{\partial})$ is of the Frobenius type if its nullsolutions $f(\underline{x})$ in $B_m(1)$ can be represented as*

$$f(\underline{x}) = \sum_{k=0}^{\infty} r^k \big\{ \alpha_k(r^2) P_k(\underline{\omega}) + r\underline{\omega}\beta_k(r^2)\widetilde{P}_k(\underline{\omega}) \big\},$$

where, for all $k \in \mathbb{N}$, the functions $\alpha_k(r^2)$ and $\beta_k(r^2)$ can be written as positive power series

$$\alpha_k(r^2) = \sum_{l=0}^{\infty} a_l r^{2l}, \quad \beta_k(r^2) = \sum_{l=0}^{\infty} b_l r^{2l}$$

converging on the open interval $]-1,1[$, with $P_k(\underline{\omega})$ and $\widetilde{P}_k(\underline{\omega})$ belonging to $M_+(k)$.

Theorem *Consider the differential operator $P(\underline{x}, \underline{\partial})$ of the Frobenius type with nullsolutions in $B_m(1)$ given by $f(\underline{x}) = \sum_{k=0}^{\infty} \big\{\alpha_k(r^2)P_k(\underline{x}) + \underline{x}\beta_k(r^2)\widetilde{P}_k(\underline{x})\big\}$, where the series converges normally on $B_m(1)$. If the conditions*:

1) $\sup_{|\underline{z}|\le 1} |\alpha_k(z)| = c_1$ *and* $\sup_{|\underline{z}|\le 1} |\beta_k(z)| = c_2$,
2) $\sum_{k=0}^{\infty} ||P_k||^2_{L_2(S^{m-1})} < \infty$ *and* $\sum_{k=0}^{\infty} ||\widetilde{P}_k||^2_{L_2(S^{m-1})} < \infty$,

are satisfied, the complexified series $f(e^{it}\underline{\omega})$ will belong to $L_2^+(LS^{m-1})$.

If the nullsolutions of the Frobenius operator $P(\underline{x}, \underline{\partial})$ satisfy the requirements of the theorem, one may consider the submodule $H \subset L_2^+(LS^{m-1})$ containing the complexified nullsolutions: $H = \ker P(\underline{z}, \underline{\partial}_z) \cap L_2^+(LS^{m-1})$. Due to the closedness of the operator $P(\underline{z}, \underline{\partial}_z)$ the submodule is also closed and its reproducing kernel, for the inner product on $L_2^+(LS^{m-1})$, can then be obtained as the projection of the Cauchy–Hua kernel $H(\underline{z}, e^{it}\underline{\omega})$ on H. The reproducing property can then be restricted to the Euclidean unit ball $B_m(1)$ leading to a reproducing kernel for the Hilbert module of nullsolutions for the operator $P(\underline{x}, \underline{\partial})$ satisfying the requirements of the theorem, for the inner product on the sphere S^{m-1} induced by the inner product on $L_2^+(LS^{m-1})$. In particular we then also have that this module is a closed submodule of $L_2(S^{m-1})$.

This will be applied in Section 5, when we construct a reproducing kernel for the function space containing nullsolutions for the hyperbolic Dirac operator.

4 Function Theory on the Klein Ball

In view of the projective nature of the definition for the Dirac operator on the hyperbolic unit ball, the Dirac operator on the Klein model can easily be obtained as follows: Putting $F(T, \underline{X}) = \lambda^\alpha F(\underline{x})$ with $\lambda = T$ and $\underline{x} = \frac{\underline{X}}{T} \in B_m(1)$, and choosing $(\lambda, \underline{x})$ as new coordinates on the FC, the projection of ∂_X acting on α-homogeneous sections becomes the operator $D_\alpha(\underline{x})$, defined by $D_\alpha(\underline{x}) = \underline{\partial} + \epsilon(r\partial_r - \alpha)$, acting on functions $f(\underline{x})$ defined on $B_m(1)$. Here, $\underline{\partial}$ and $r\partial_r$, respectively, stand for the Dirac operator and the Euler operator on $\mathbb{R}^m$. Note that $D_\alpha(\underline{x})$ is of the form $P(\underline{x}, \underline{\partial})$. Later, it will be shown that under certain restrictions on α it is an operator of the Frobenius type.

In reference [16] the author has proved a theorem to construct hyperbolic monogenics on the Klein ball. For this purpose, let us first define:

Definition 1 *For an open subset $\Omega_K \subset B_m(1)$ and an arbitrary $\alpha \in \mathbb{C}$ we define the set of hyperbolic monogenics in Ω_K as $\mathscr{H}_K^\alpha(\Omega_K) = \{f \in C^1(\Omega_K) : D_\alpha(\underline{x})f = 0\}$.*

Definition 2 *For an arbitrary complex α and an arbitrary integer l, we put*

$$\mathrm{Mod}(\alpha, l, \underline{x}) = F_1^{(l)}(|\underline{x}|^2) + \frac{l-\alpha}{2l+m}\, \underline{x}\epsilon F_2^{(l)}(|\underline{x}|^2),$$

with

$$F_1^{(l)}(t) = F\left(\frac{1+l-\alpha}{2}, \frac{l-\alpha}{2}; l+\frac{m}{2}; t\right) \quad F_2^{(l)}(t) = F\left(\frac{1+l-\alpha}{2}, 1+\frac{l-\alpha}{2}; 1+l+\frac{m}{2}; t\right).$$

The following Theorems then hold:

Theorem 1 *Let $P_k(\underline{\xi}) \in M_+(k)$ and $\alpha \in \mathbb{C}$. We then have*

$$\mathscr{P}_\alpha(\underline{x}) = \mathrm{Mod}(\alpha, k, \underline{x})P_k(\underline{x}) \in \mathscr{H}_K^\alpha(B_m(1)).$$

Theorem 2 *Let $Q_k(\underline{\xi}) \in M_-(k)$ and $\alpha \in \mathbb{C}$ such that $\alpha + m \notin -k - \mathbb{N}$. We then have*

$$\mathscr{Q}_\alpha(\underline{x}) = \mathrm{Mod}(\alpha, 1-k-m, \underline{x})Q_k(\underline{x}) \in \mathscr{H}_K^\alpha(B_m(1) \setminus \{\underline{0}\}).$$

These theorems allow us to derive the most important function-theoretical results for the Dirac operator on the Klein ball from their counterparts in flat Euclidean space by *modulation* (i.e. multiplication with the appropriate modulation factor). We give an overview of the most important results, for which we refer to a recent series of papers (see e.g. [6], [7]).

The fundamental solution for the Dirac operator on the Klein ball is easily found as

$$E_\alpha(\underline{x}) \quad = \quad \mathrm{Mod}(\alpha, 1-m; \underline{x})E(\underline{x}),$$

and satisfies $D_\alpha(\underline{x})E_\alpha(\underline{x}) = -\delta(\underline{x})$. To obtain a fundamental solution $E_\alpha(\underline{x}, \underline{y})$ with singularity for $\underline{x} = \underline{y}$ one can *not* merely translate the argument, since the operator $D_\alpha(\underline{x})$ is not invariant under translations. Instead, one has to reconsider the projective picture, apply a Lorentz transformation and project back onto the Klein model. This gives rise to a fundamental solution $E_\alpha(\underline{x}, \underline{y})$ satisfying $D_\alpha(\underline{x})E_\alpha(\underline{x}, \underline{y}) = -\delta(\underline{x} - \underline{y}) = E_\alpha(\underline{x}, \underline{y})D_\beta(\underline{y})$, where $\alpha + \beta + m = 0$. A series expansion for $E_\alpha(\underline{x}, \underline{y})$ can easily be obtained by modulation of the classical decomposition for the Cauchy kernel on $\mathbb{R}_m$ (see e.g. reference [2]):

$$E(\underline{y} - \underline{x}) = -\frac{1}{A_m}\sum_{k=0}^{\infty} \frac{|\underline{y}|^k}{|\underline{x}|^{k+m-1}}\left\{C_k^{\frac{m}{2}}(t)\underline{\xi} - C_{k-1}^{\frac{m}{2}}(t)\underline{\eta}\right\} = -\frac{1}{A_m}\sum_{k=0}^{\infty} \frac{|\underline{y}|^k}{|\underline{x}|^{k+m-1}} C_k(\underline{\xi}, \underline{\eta}),$$

where $\underline{x} = |\underline{x}|\underline{\xi}$ and $\underline{y} = |\underline{y}|\underline{\eta}$. This series converges for $|\underline{y}| < |\underline{x}|$. Because each term of the series yields an inner spherical monogenic in the $\underline{y}$-variable and an outer spherical monogenic in the $\underline{x}$-variable, Theorems 1 and 2 can be used to modulate $E(\underline{y} - \underline{x})$. We then obtain

$$E_\alpha(\underline{y}, \underline{x}) = \frac{1}{A_m}\sum_{k=0}^{\infty} E_\alpha^{(k)}(\underline{y}, \underline{x}) = -\frac{1}{A_m}\sum_{k=0}^{\infty} \mathrm{Mod}(\alpha, k; \underline{y})\frac{|\underline{y}|^k C_k(\underline{\xi}, \underline{\eta})}{|\underline{x}|^{k+m-1}}\overline{\mathrm{Mod}}(\beta, 1-k-m; \underline{x}).$$

Consider then an open subset Ω_K in $B_m(1)$, and a compact $C_K \subset \Omega_K$ with smooth boundary and interior $in(C_K)$. We have proved:

Theorem (Cauchy) *Let $f \in \mathscr{H}_K^\alpha(\Omega_K)$. Then, for all $\underline{y} \in in(C_K)$, we have*

$$\int_{\partial C_K} E_\alpha(\underline{y}, \underline{x})\sigma_K(\underline{x}, d\underline{x})f(\underline{x}) \quad = \quad f(\underline{y}),$$

with $\sigma_K(\underline{x}, d\underline{x})$ the oriented surface element on the Klein ball.

Theorem (Taylor) *Let $f(\underline{y}) \in \mathscr{H}_K^\alpha\big(B_m(r)\big)$, $\alpha \in \mathbb{C}$ and $\alpha + m \notin -\mathbb{N}$. There exists $(f^{(k)}(\underline{y}))_{k\in\mathbb{N}}$ with $\underline{y} \mapsto f^{(k)}(\underline{y})$ belonging to $\mathscr{H}_K^\alpha\big(B_m(1)\big)$ for each k, such that $f(\underline{y}) = \sum_{k=0}^\infty f^{(k)}(\underline{y})$. The Taylor expansion for $f(\underline{y}) \in \mathscr{H}_K^\alpha\big(B_m(r)\big)$ converges normally on compact sets $\overline{B}_m(\rho)$, with $|\underline{y}| \le \rho < r$. An explicit expression for $f^{(k)}(\underline{y})$ is given by*

$$f^{(k)}(\underline{y}) \quad = \quad \mathrm{Mod}(\alpha, k; \underline{y})\frac{|\underline{y}|^k}{r^k}P(k)[\mathrm{Mod}(\beta, 1-k-m; r\underline{\xi})(r\underline{\xi}\epsilon - 1)f(r\underline{\xi})](\underline{\eta}),$$

where $P(k)[f]$ denotes the projection of a function $f \in L_2(S^{m-1})$ onto the space of inner spherical monogenics.

The latter theorem will often be used in the following form: For $f(\underline{y}) \in \mathscr{H}_K^\alpha\big(B_m(1)\big)$ there exists a sequence $(P_k(\underline{\eta}))_k$ of inner spherical monogenics on S^{m-1} such that $f(\underline{y}) = \sum_k \mathrm{Mod}(\alpha, k; \underline{y})P_k(\underline{y})$.

5 Hyperbolic Boundary Values

We begin this section by introducing a new function space $\mathscr{M}L_2^\alpha(S^{m-1})$ and constructing examples by means of a Cauchy-type integral transform:

Definition 3 *Let α be an arbitrary complex number. We put*

$$\mathscr{M}L_2^\alpha(S^{m-1}) = \left\{f \in L_2(S^{m-1}) : f(\underline{\omega}) = \lim_{r\to 1} f^*(\underline{x}),\ f^* \in \mathscr{H}_K^\alpha\big(B_m(1)\big)\right\},$$

i.e. $\mathscr{M}L_2^\alpha(S^{m-1})$ contains those functions in $L_2(S^{m-1})$ which are the radial limits (in the L_2-sense) of hyperbolic monogenics on the Klein ball.

In a previous paper (see [17]) we have defined the so-called *photogenic Cauchy transform* of a function $f \in L_2(S^{m-1})$ as

$$\mathscr{C}_P^\alpha[f](\underline{x}) \quad = \quad \frac{1}{A_m}\int_{S^{m-1}} \mathscr{F}_\alpha(\underline{x},\underline{\omega})\underline{\omega} f(\underline{\omega}) dS(\underline{\omega}),$$

where $\mathscr{F}_\alpha(\underline{x},\underline{\omega})$ stands for the photogenic Cauchy kernel, the fundamental solution for the Dirac operator on the hyperbolic Klein ball with singularity on the boundary:

$$D_\alpha(\underline{x})\mathscr{F}_\alpha(\underline{x},\underline{\omega}) \quad = \quad -\delta(\underline{x}-\underline{\omega}),\ \underline{\omega} \in S^{m-1}.$$

Note that $\mathscr{C}_P^\alpha[f](\underline{x}) \in \mathscr{H}_K^\alpha\big(B_m(1)\big)$ for all $\alpha \in \mathbb{C}$, whenever it is defined. In the very same paper, the photogenic Cauchy transform of inner and outer spherical monogenics was determined. It was found that

$$\begin{aligned}\mathscr{C}_P^\alpha[P_k](\underline{x}) \quad &= \quad c_{\alpha,m}\frac{\Gamma(\alpha+m+k+1)}{2^k\Gamma\left(k+\frac{m}{2}\right)}\mathrm{Mod}(\alpha,k;\underline{x})P_k(\underline{x}),\\ \mathscr{C}_P^\alpha[Q_k](\underline{x}) \quad &= \quad -c_{\alpha,m}\frac{(1+\alpha-k)\Gamma(\alpha+m+k)}{2^k\Gamma\left(k+\frac{m}{2}\right)}\mathrm{Mod}(\alpha,k;\underline{x})P_k(\underline{x})\epsilon, \qquad (1)\end{aligned}$$

with $c_{\alpha,m}$ a constant depending on α and m only. Defining the boundary values respectively as the radial limits $\mathscr{C}_P^\alpha[P_k]\uparrow(\underline{\xi}) = \lim_{r\to 1-}\mathscr{C}_P^\alpha[P_k](r\underline{\xi})$, $\mathscr{C}_P^\alpha[Q_k]\uparrow(\underline{\xi}) = \lim_{r\to 1-}\mathscr{C}_P^\alpha[Q_k](r\underline{\xi})$, we have found for arbitrary α such that $\mathrm{Re}(\alpha) > \frac{1-m}{2}$:

$$\mathscr{C}_P^\alpha[P_k]\uparrow(\underline{\xi}) = \mathscr{P}_\alpha(\Gamma)P_k(\underline{\xi}), \quad \mathscr{C}_P^\alpha[Q_k]\uparrow(\underline{\xi}) = \mathscr{P}_\alpha(\Gamma)Q_k(\underline{\xi}),$$

where $\mathscr{P}_\alpha(\Gamma)$ stands for the second-order differential operator on the sphere defined by

$$\mathscr{P}_\alpha(\Gamma) \quad = \quad \frac{\Gamma\left(\frac{m-1}{2}\right)}{8\pi^{\frac{m-1}{2}}}\frac{((1+\beta+\Gamma)+\underline{\xi}\epsilon(\Gamma+\alpha))(\beta+\Gamma)}{\left(\alpha+\frac{m-1}{2}\right)\left(\alpha+\frac{m+1}{2}\right)},$$

with $\alpha+\beta+m=0$. Note that for spherical monogenics $P_k(\underline{\xi})$ and $Q_k(\underline{\xi})$ we get immediately that $\mathscr{C}_P^\alpha[P_k]\uparrow$ and $\mathscr{C}_P^\alpha[Q_k]\uparrow$ belong to $\mathscr{M}L_2^\alpha(S^{m-1})$.

On the other hand we have a mapping from the Sobolev space $W_2(S^{m-1})$, defined by

$$W_2(S^{m-1}) = \{f : L_{\underline{\sigma\tau}} f \in L_2(S^{m-1}),\ |\underline{\sigma}| = |\underline{\tau}| \leq 2\},$$

where $L_{\underline{\sigma\tau}} = L_{\sigma_1\tau_1}\cdots L_{\sigma_n\tau_n}$ for multi-indices $\underline{\sigma}$ and $\underline{\tau} \in \mathbb{N}^n$, into the space of L_2-boundary values of hyperbolic monogenics. This mapping is denoted by $\mathscr{C}_P^\alpha[\cdot]\uparrow$ and is defined by

$$\mathscr{C}_P^\alpha[\cdot]\uparrow\ \ W_2(S^{m-1}) \ \mapsto\ \mathscr{M}L_2^\alpha(S^{m-1}),\ \ f(\underline{\xi}) \ \mapsto\ \lim_{r\to 1-}\mathscr{C}_P^\alpha[f](r\underline{\xi}).$$

This mapping is, however, *not* one-to-one, since it has a non-trivial kernel. This can easily be seen as follows: In view of formula (1) it is clear that

$$\mathscr{C}_P^\alpha[(1+\alpha-k)P_k + (\alpha+m+k)\underline{\omega}\epsilon P_k] \quad = \quad 0,$$

which, by means of the fact that $P_k(\underline{\omega})$ and $Q_k(\underline{\omega})$ are eigenfunctions for the spherical Dirac operator Γ with resp. eigenvalues $(-k)$ and $(k+m-1)$, can be rewritten as

$$\mathscr{C}_P^\alpha[(\Gamma+1+\alpha)(1+\underline{\omega}\epsilon)P_k] \quad = \quad 0.$$

Recalling the explicit definition for the photogenic Cauchy transform, and using the fact that $(1+\underline{\omega}\epsilon) = (1+\underline{\omega}\epsilon)\underline{\omega}\epsilon$, we thus get that, for all $k \in \mathbb{N}$,

$$\int_{S^{m-1}} \mathscr{F}_\alpha(\underline{x},\underline{\omega})\underline{\omega}(\Gamma+1+\alpha)(1+\underline{\omega}\epsilon)P_k(\underline{\omega})dS(\underline{\omega}) \quad = \quad 0$$

and also

$$\int_{S^{m-1}} \mathscr{F}_\alpha(\underline{x},\underline{\omega})\underline{\omega}(\Gamma+1+\alpha)(1+\underline{\omega}\epsilon)Q_k(\underline{\omega})dS(\underline{\omega}) \quad = \quad 0.$$

Both integrals are indeed vanishing because the photogenic Cauchy kernel satisfies the following differential equation *in* $\underline{\omega}$ on the unit ball $B_m(1)$:

$$\{\Gamma(\underline{\omega}+\epsilon)+(1+\alpha)(\underline{\omega}-\epsilon)-(m-1)\epsilon\}\mathscr{F}_\alpha(\underline{x},\underline{\omega})=0.$$

Eventually this means that

$$\ker\mathscr{C}_P^\alpha[\cdot]\uparrow=\Big\{f\in W_2(S^{m-1}):f=(\Gamma+1+\alpha)\sum_k c_k(1+\underline{\omega}\epsilon)P_k(\underline{\omega})\Big\}.$$

In view of the fact that the operator $\mathscr{C}_P^\alpha[\cdot]\uparrow$ is continuous for $\mathrm{Re}(\alpha)>\frac{1-m}{2}$, its kernel is closed and therefore the domain of the mapping $\mathscr{C}_P^\alpha[\cdot]\uparrow$ can be defined as the orthogonal complement of the kernel.

Note that in our paper [17] we have also given a meaning to the photogenic Cauchy transform as an integral transform acting on distributions.

Consider then an arbitrary element $f\in\mathscr{M}L_2^\alpha(S^{m-1})$. By definition, we have $f(\underline{\xi})=\lim_{r\to 1-}f(r\underline{\xi})$, $f(\underline{x})\in\mathscr{H}_K^\alpha\big(B_m(1)\big)$. As $f(\underline{x})\in\mathscr{H}_K^\alpha\big(B_m(1)\big)$, it follows from Stokes' theorem that there exist inner spherical monogenics such that $f(\underline{x})=\sum_k\mathrm{Mod}(\alpha,k;\underline{x})P_k(\underline{x})$. Hence, for complex α with $\mathrm{Re}(\alpha)>\frac{1-m}{2}$ we get $f(\underline{\xi})=\sum_k\mathrm{Mod}(\alpha,k;\underline{\xi})P_k(\underline{\xi})$.

Remark This argument also shows that the function spaces $\mathscr{M}L_2^\alpha(S^{m-1})$ are trivial for $\mathrm{Re}(\alpha)\leq\frac{1-m}{2}$.

Note that the inverse only holds if $\alpha\in\mathbb{R}$. In other words, for *real* $\alpha>\frac{1-m}{2}$ we get

$$f(\underline{\xi})\in\mathscr{M}L_2^\alpha(S^{m-1})\iff f(\underline{\xi})=\sum_k\mathrm{Mod}(\alpha,k;\underline{\xi})P_k(\underline{\xi}).$$

For the proof of the inverse implication it suffices to put $\widetilde{f}(\underline{x})=\sum_k\mathrm{Mod}(\alpha,k;\underline{x})P_k(\underline{x})$, from which it immediately follows that $f(\underline{\xi})=\lim_{r\to 1}\widetilde{f}(\underline{x})$. The series converges to $\widetilde{f}(\underline{x})$ for the supremum norm on the unit ball $B_m(1)$, which follows from an estimate derived a bit further in the paper.

The aim is to prove that under certain restrictions on α, the function $f(e^{it}\underline{\xi})$ belongs to the Hilbert module $L_2^+(LS^{m-1})$. The module $\mathscr{M}L_2^\alpha(S^{m-1})$ will then have a reproducing kernel, obtained by projection of the Cauchy–Hua kernel, because the operator $D_\alpha(\underline{x})$ is a Frobenius operator satisfying the additional requirements under certain conditions on α. First of all note that the function $f(\underline{z})$ is holomorphic in the Lie ball, which follows from Siciak's theorem. To examine the boundary behaviour of this holomorphic function, it thus suffices to consider the extension of $f(\underline{\xi})$ to the Lie sphere: $f(e^{it}\underline{\xi})=\sum_k\mathrm{Mod}(\alpha,k;e^{it}\underline{\xi})P_k(e^{it}\underline{\xi})$. By definition we have $f(e^{it}\underline{\xi})\in L_2(LS^{m-1})$ iff $\|f(e^{it}\underline{\xi})\|_{L_2(LS^{m-1})}<\infty$, where the Lie norm is given by

$$\frac{1}{\pi}\sum_{k=0}^{\infty}\bigg\{\int_0^\pi|F_1^{(k)}(e^{2it})|^2dt-\bigg|\frac{k-\alpha}{2k+m}\bigg|\int_0^\pi|F_2^{(k)}(e^{2it})|^2dt\bigg\}\|P_k\|^2_{L_2(S^{m-1})},$$

whence $f(e^{it}\underline{\xi})\in L_2(LS^{m-1})$ if both

$$\frac{1}{\pi}\bigg(\int_0^\pi|F_1^{(k)}(e^{2it})|^2dt\|P_k\|^2_{L_2(S^{m-1})}\bigg)_k\in l_1\quad\text{and}\quad\frac{1}{\pi}\bigg(\int_0^\pi|F_2^{(k)}(e^{2it})|^2dt\|P_k\|^2_{L_2(S^{m-1})}\bigg)_k\in l_1.$$

In view of the fact that $f(\underline{\xi})=\sum_k\mathrm{Mod}(\alpha,k;\underline{\xi})P_k(\underline{\xi})\in L_2(S^{m-1})$, whence both $\big(|F_1^{(k)}(1)|^2\|P_k\|^2\big)_k$ and $\big(|F_2^{(k)}(1)|^2\|P_k\|^2\big)_k$ belong to l_1, it suffices to find conditions on α such that, for all $k\geq k_0$, we have: $\frac{1}{\pi}\int_0^\pi|F_j^{(k)}(e^{2it})|^2dt\leq|F_j^{(k)}(1)|^2,\ j\in\{1,2\}$. To do so, we use the following: Let a, b and c be *real* such that $c>b>0$, $a>0$ and $c-a-b>0$. We then have Euler's integral representation formula for the hypergeometric function $F(a,b;c;e^{ix})$:

$$F(a,b;c;e^{ix})\;=\;\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-te^{ix})^{-a}dt.$$

From this, it easily follows that

$$\big|F(a,b;c;e^{ix})\big|\;\leq\;\bigg|\frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\bigg|\int_0^1\big|t^{b-1}(1-t)^{c-b-1}\big|\,\big|(1-te^{ix})^{-a}\big|\,dt,$$

which, in view of the fact that the parameters are real, reduces to

$$\left|F(a,b;c;e^{ix})\right| \quad \leq \quad \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}\left|(1-te^{ix})\right|^{-a}dt.$$

As $a>0$ and $\left|1-te^{ix}\right| = \left((1+t^2)-2t\cos x\right)^{\frac{1}{2}} \geq 1-t$, we eventually find that

$$\left|F(a,b;c;e^{ix})\right| \leq \frac{\Gamma(c)}{\Gamma(c-b)\Gamma(b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-t)^{-a}dt = F(a,b;c;1).$$

In the present situation this means that for *real* α such that $\alpha > \frac{1-m}{2}$, we have that $f \in \mathscr{M}L_2^\alpha(S^{m-1}) \Longrightarrow f \in L_2^+(LS^{m-1})$. This means that for these α the operator $D_\alpha(\underline{x})$ is a Frobenius operator satisfying the requirements of the theorem mentioned at the end of Section 3.3, from which it follows that a reproducing kernel can be constructed. Since $f(\underline{z}) \in L_2^+(LS^{m-1})$, we get $f(\underline{z}) = \frac{1}{A_m\pi}\int_0^\pi\int_{S^{m-1}} H^+(\underline{z}, e^{it}\underline{\omega})f(e^{it}\underline{\omega})dS(\underline{\omega})dt$. Using the explicit formula for the Cauchy–Hua kernel, we then get

$$f(r\underline{\xi}) \quad = \quad \frac{1}{A_m\pi}\sum_{k,l=0}^{\infty}\int_0^\pi\int_{S^{m-1}}(r\underline{\xi})^l r^k e^{-ikt}C_k^*(\underline{\xi},\underline{\omega})(e^{it}\underline{\omega})^{-l}f(e^{it}\underline{\omega})dS(\underline{\omega})dt,$$

where $\theta = \langle\underline{\xi},\underline{\omega}\rangle$ and $C_k^*(\underline{\xi},\underline{\omega}) = C_k^{\frac{m}{2}}(\theta) + \underline{\xi}\underline{\omega}C_{k-1}^{\frac{m}{2}}(\theta) = -\underline{\xi}C_k^*(\underline{\xi},\underline{\omega})$ with $\underline{x} = r\underline{\xi} \in B_m(1)$. With $f(\underline{x}) = \sum_{q=0}^\infty \mathrm{Mod}(\alpha,q;\underline{x})P_q(\underline{x})$ we get for $f(r\underline{\xi})$, for all $\underline{x} \in B_m(1)$

$$\sum_{q,k,l=0}^{\infty}\frac{r^{k+l}\underline{\xi}^l}{A_m\pi}\int_0^\pi\int_{S^{m-1}} e^{i(q-k-l)t}C_k^*(\underline{\xi},\underline{\omega})\underline{\omega}^{-l}\mathrm{Mod}(\alpha,q;e^{it}\underline{\omega})P_q(\underline{\omega})dS(\underline{\omega})dt.$$

In view of the definition for the modulation factor, this reduces to the sum of two terms:

$$\Sigma_1 \quad = \quad \sum_{q,k,l=0}^{\infty}\frac{r^{k+l}\underline{\xi}^l}{A_m\pi}\int_0^\pi e^{i(q-k-l)t}F_1^{(q)}(e^{2it})dt\int_{S^{m-1}} C_k^*(\underline{\xi},\underline{\omega})\underline{\omega}^{-l}P_q(\underline{\omega})dS(\underline{\omega})$$

and

$$\Sigma_2 = \sum_{q,k,l=0}^{\infty}\frac{q-\alpha}{2q+m}\frac{r^{k+l}\underline{\xi}^l}{A_m\pi}\int_0^\pi e^{i(1+q-k-l)t}F_2^{(q)}(e^{2it})dt\int_{S^{m-1}} C_k^*(\underline{\xi},\underline{\omega})\underline{\omega}^{-l}\underline{\omega}\epsilon P_q(\underline{\omega})dS(\underline{\omega}).$$

Consider for example Σ_1. Due to the orthogonality of spherical monogenics on the sphere and the fact that $C_k^*(\underline{\xi},\underline{\omega}) = \overline{C_k^*(\underline{\omega},\underline{\xi})}$, with $C_k^*(\underline{\omega},\underline{\xi})$ an inner spherical monogenic of order k, the integral in $\underline{\omega}$ differs from zero only if both $l \in 2\mathbb{N}$ and $q=k$. Hence,

$$\Sigma_1 \quad = \quad \sum_{k,l=0}^{\infty}\frac{r^{k+2l}}{A_m\pi}\int_0^\pi e^{-2ilt}F_1^{(k)}(e^{2it})dt\int_{S^{m-1}} C_k(\underline{\xi},\underline{\omega})^* P_k(\underline{\omega})dS(\underline{\omega}).$$

As

$$\int_0^\pi e^{-2ilt}F_1^{(k)}(e^{2it})dt \quad = \quad \sum_{j=0}^{\infty}\frac{\left(\frac{k-\alpha}{2}\right)_j\left(\frac{1+k-\alpha}{2}\right)_j}{j!\left(k+\frac{m}{2}\right)_j}\int_0^\pi e^{-2i(j-l)t}dt,$$

the sumation in l disappears, and only the term for which $j=l$ remains. This leads to

$$\Sigma_1 \quad = \quad \sum_{k=0}^{\infty}\frac{r^k}{A_m}F_1^{(k)}(r^2)\int_{S^{m-1}} C_k^*(\underline{\xi},\underline{\omega})P_k(\underline{\omega})dS(\underline{\omega}).$$

In a completely similar way, we arrive at

$$\Sigma_2 \quad = \quad \sum_{k=0}^{\infty}\frac{k-\alpha}{2k+m}\frac{r^{1+k}}{A_m}F_2^{(k)}(r^2)\underline{\xi}\epsilon\int_{S^{m-1}} C_k^*(\underline{\xi},\underline{\omega})P_k(\underline{\omega})dS(\underline{\omega}),$$

from which it then immediately follows that $f(r\underline{\xi}) = \sum_{k=0}^\infty \mathrm{Mod}(\alpha,k;r\underline{\xi})P_k(r\underline{\xi})$, as is to be expected!

We then propose the following form for the reproducing kernel $K_\alpha(\underline{x},\underline{\omega})$ for the module $\mathscr{M}L_2^\alpha(S^{m-1})$, in the case $\alpha > \frac{1-m}{2}$:

$$K_\alpha(\underline{\omega},\underline{x}) \quad = \quad \sum_{k=0}^{\infty} r^k\frac{\mathrm{Mod}(\alpha,k;r\underline{\xi})C_k^*(\underline{\xi},\underline{\omega})\overline{\mathrm{Mod}(\alpha,k;\underline{\omega})}}{|F_1^{(k)}(1)|^2 - |F_2^{(k)}(1)|^2}.$$

This kernel satisfies the necessary conditions.

In view of its very own definition, we have $K_\alpha(\underline{x}, \underline{y}) = \overline{K_\alpha(\underline{y}, \underline{x})}$, which is the property of anti-symmetry.

The kernel $K_\alpha(\underline{\xi}, \underline{\omega})$ belongs to the module $\mathscr{M}L_2^\alpha(S^{m-1})$.

It has the reproducing property:

$$\int_{S^{m-1}} \overline{K_\alpha(r\underline{\xi}, \underline{\omega})} f(\underline{\omega}) dS(\underline{\omega}) = \sum_{l=0}^{\infty} \int_{S^{m-1}} \overline{K_\alpha(r\underline{\xi}, \underline{\omega})} \mathrm{Mod}(\alpha, l; \underline{\omega}) P_l(\underline{\omega}) dS(\underline{\omega}),$$

which by means of the orthogonality of spherical monogenics on the sphere reduces to

$$\int_{S^{m-1}} \overline{K_\alpha(r\underline{\xi}, \underline{\omega})} f(\underline{\omega}) dS(\underline{\omega}) \quad = \quad f(r\underline{\xi}).$$

This leads to

Theorem *For real α such that $\alpha > \frac{1-m}{2}$, the space $\mathscr{M}L_2^\alpha(S^{m-1})$ is a Hilbert module with reproducing kernel, given by $K_\alpha(\underline{\omega}, \underline{\xi}) = \sum_{k=0}^{\infty} \frac{\mathrm{Mod}(\alpha,k;\underline{\xi}) C_k(\underline{\xi},\underline{\omega}) \overline{\mathrm{Mod}(\alpha,k;\underline{\omega})}}{|F_1^{(k)}(1)|^2 - |F_2^{(k)}(1)|^2}$.*

Remark Note that for $\alpha = k$ the reproducing kernel $K_k(\underline{\omega}, \underline{\xi})$ reduces to the classical Szego kernel $K_k(\underline{\omega}, \underline{\xi}) = \sum_{k=0}^{\infty} C_k^*(\underline{\xi}, \underline{\omega}) = \frac{1+\underline{\xi}\underline{\omega}}{|1+\underline{\xi}\underline{\omega}|^m}$. Hence, the reproducing kernel for the space of hyperbolic monogenics on the Klein ball can again be found as a modulation of the "classical" kernel for the Dirac operator on the flat Euclidean space $\mathbb{R}^m$, albeit only under certain restrictions on the parameter α.

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